

Complexity Theory

Part Two

Recap from Last Time

The Complexity Class **P**

- The complexity class **P** (***polynomial time***) is defined as

$$\mathbf{P} = \{ L \mid \text{There is a polynomial-time decider for } L \}$$

- Intuitively, **P** contains all decision problems that can be solved efficiently.
- This is like class **R**, except with “efficiently” tacked onto the end.

The Complexity Class **NP**

- The complexity class **NP** (*nondeterministic polynomial time*) contains all problems that can be verified in polynomial time.
- Formally:

$$\mathbf{NP} = \{ L \mid \text{There is a polynomial-time verifier for } L \}$$

- Intuitively, **NP** is the set of problems where “yes” answers can be checked efficiently.
- This is like the class **RE**, but with “efficiently” tacked on to the definition.

The Biggest Unsolved Problem in
Theoretical Computer Science:

$$\mathbf{P} \stackrel{?}{=} \mathbf{NP}$$

P = { L | there is a polynomial-time
decider for L }

NP = { L | there is a polynomial-time
verifier for L }

R = { L | there is a ~~polynomial-time~~
decider for L }

RE = { L | there is a ~~polynomial-time~~
verifier for L }

We know that $\mathbf{R} \neq \mathbf{RE}$.

So does that mean $\mathbf{P} \neq \mathbf{NP}$?

A Problem

- The **R** and **RE** languages correspond to problems that can be decided and verified, *period*, without any time bounds.
- To reason about what's in **R** and what's in **RE**, we used two key techniques:
 - **Universality**: TMs can simulate other TMs.
 - **Self-Reference**: TMs can get their own source code.
- Why can't we just do that for **P** and **NP**?

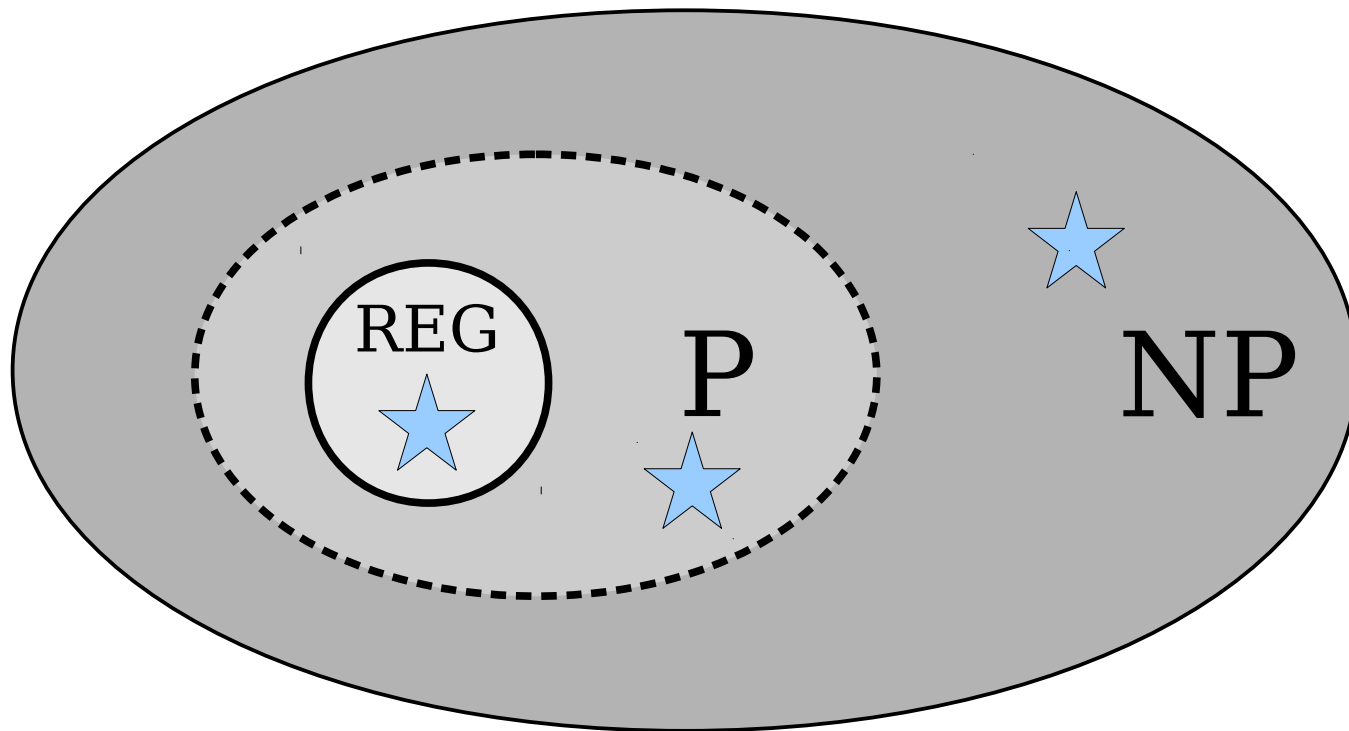
Theorem (Baker-Gill-Solovay): Any proof that purely relies on universality and self-reference cannot resolve $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

Proof: Take CS154!

So how *are* we going to
reason about **P** and **NP**?

New Stuff!

A Challenge



Problems in **NP** vary widely in their difficulty, even if **P = NP**.

How can we rank the relative difficulties of problems?

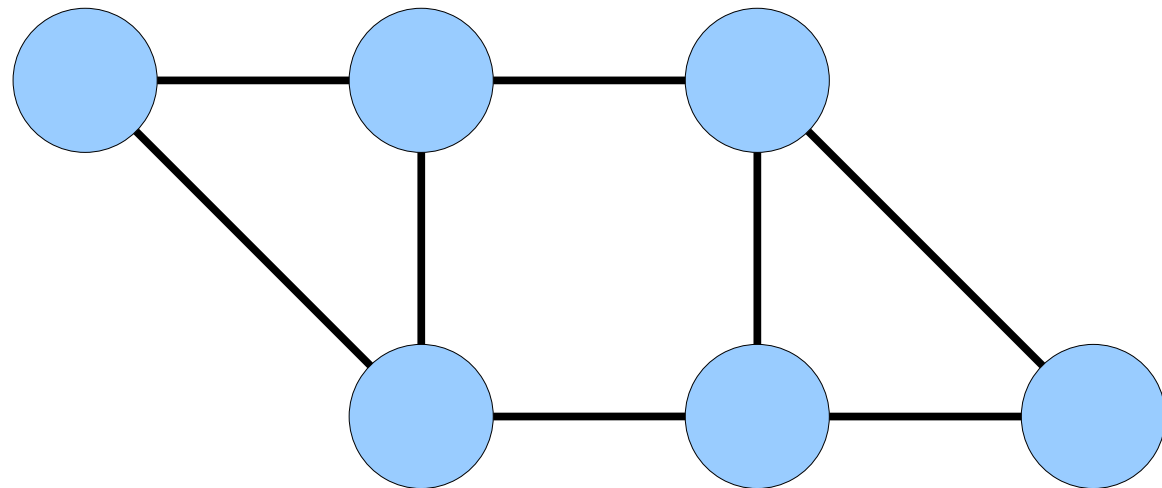
Reducibility

Maximum Matching

- Given an undirected graph G , a **matching** in G is a set of edges such that no two edges share an endpoint.
- A **maximum matching** is a matching with the largest number of edges.

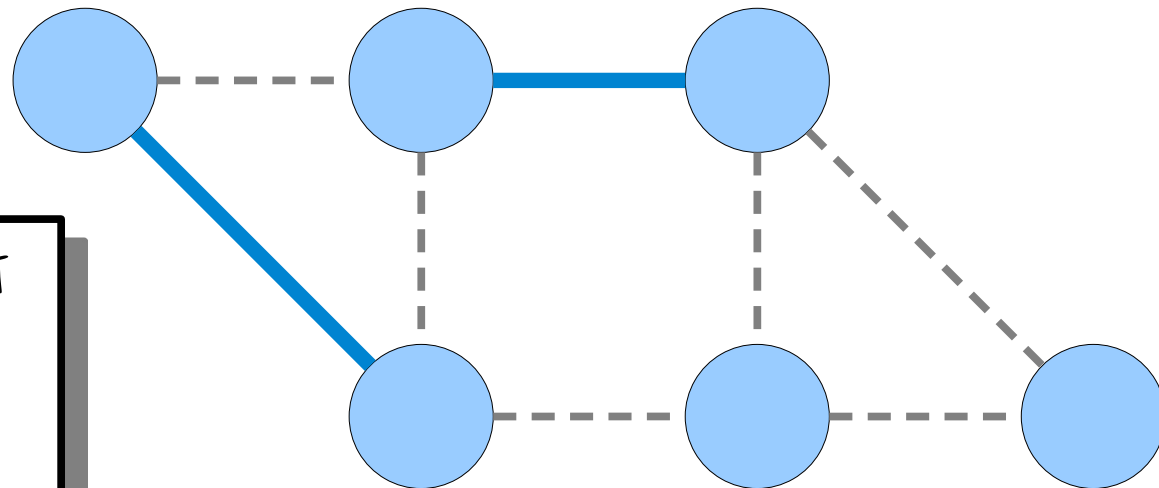
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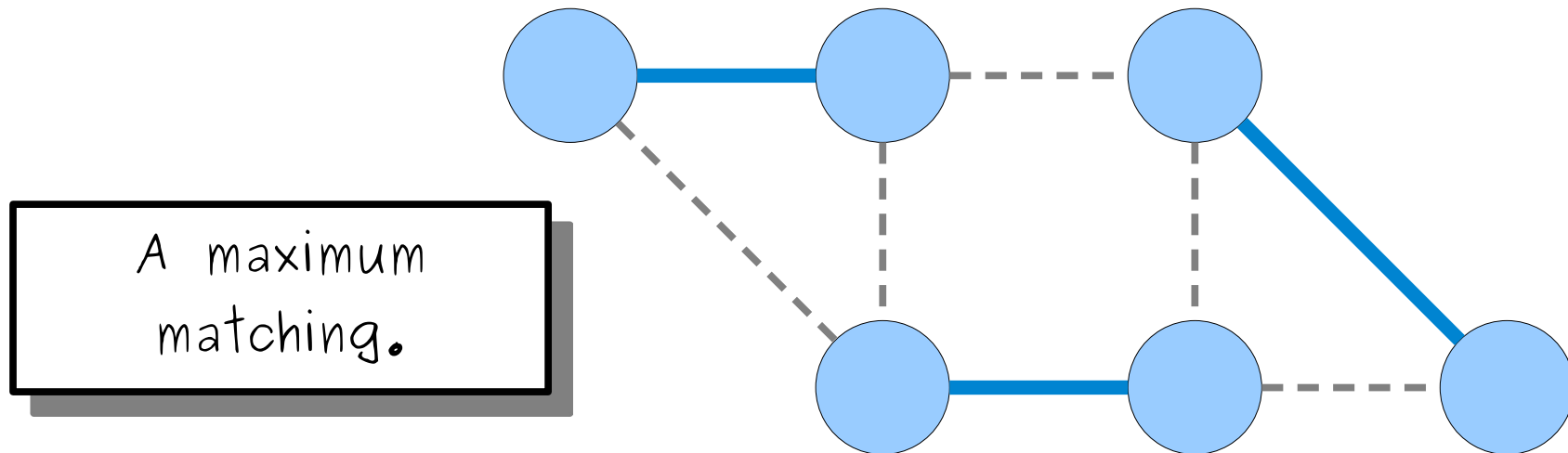
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A matching, but
not a maximum
matching.

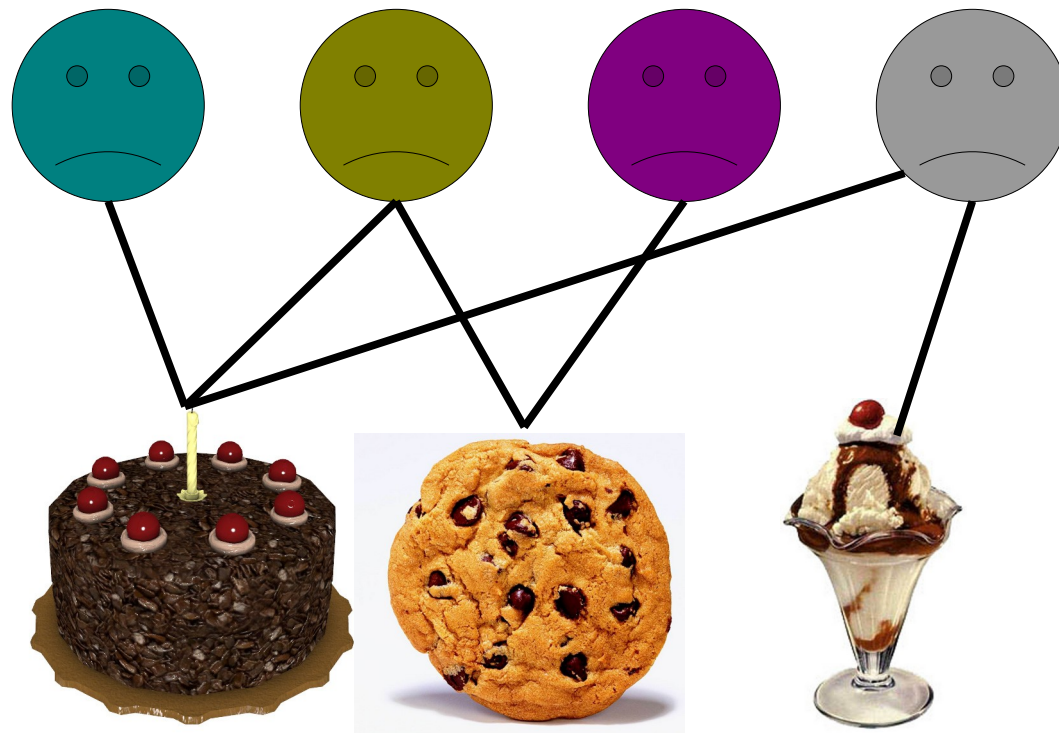
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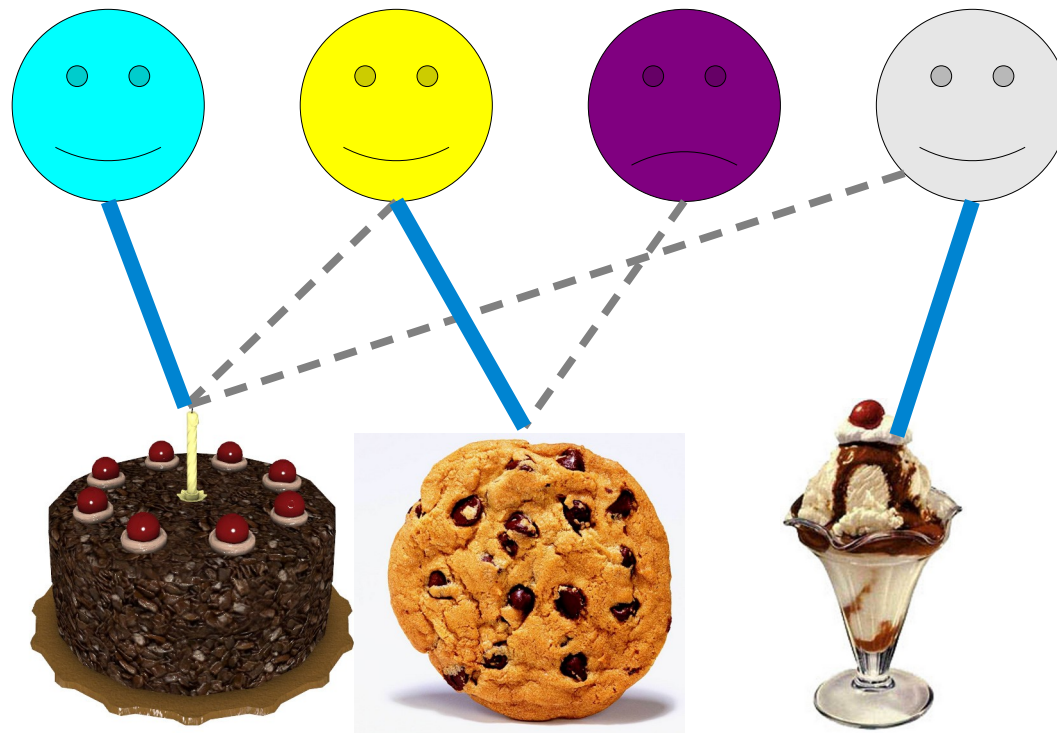
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Maximum Matching

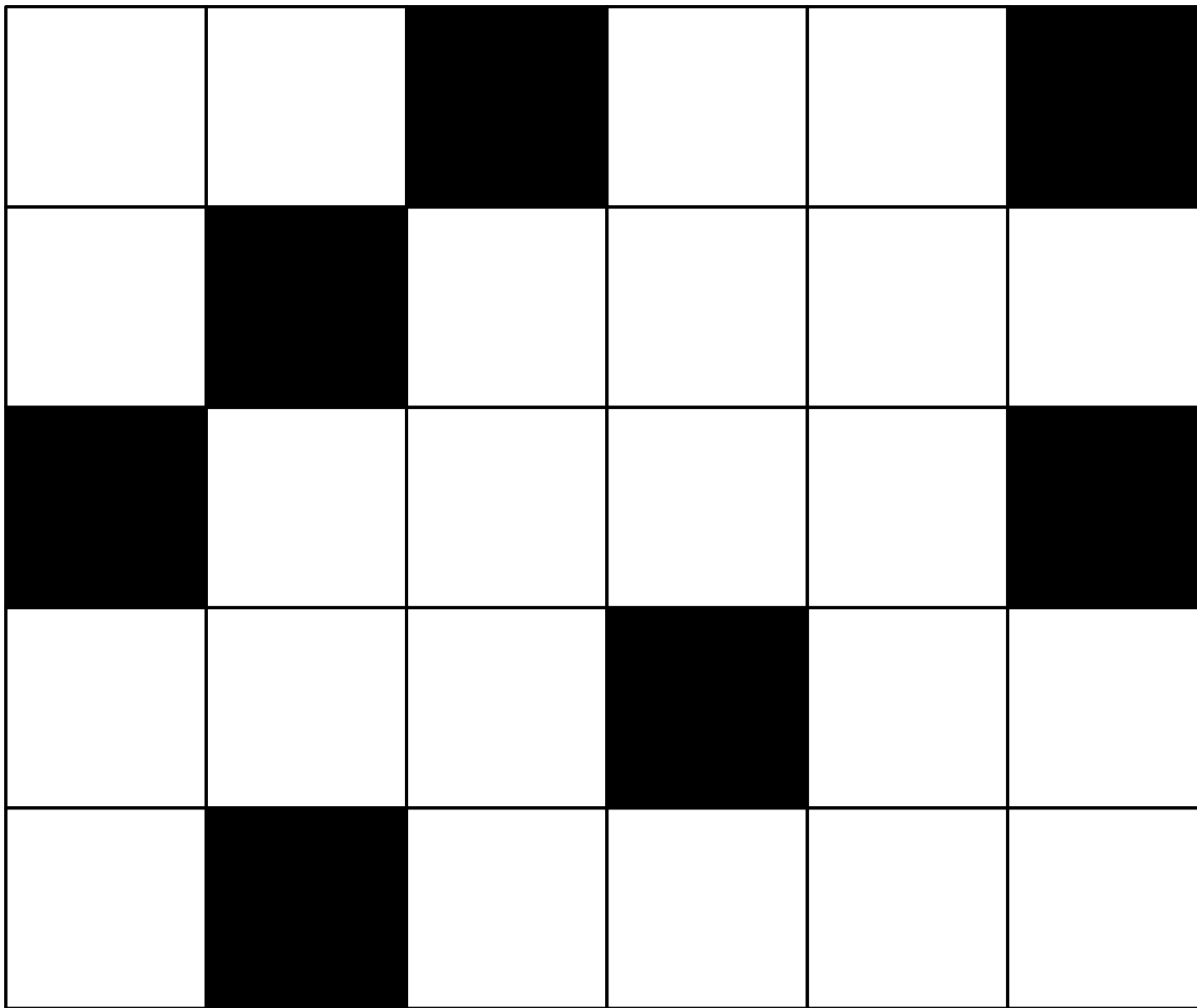
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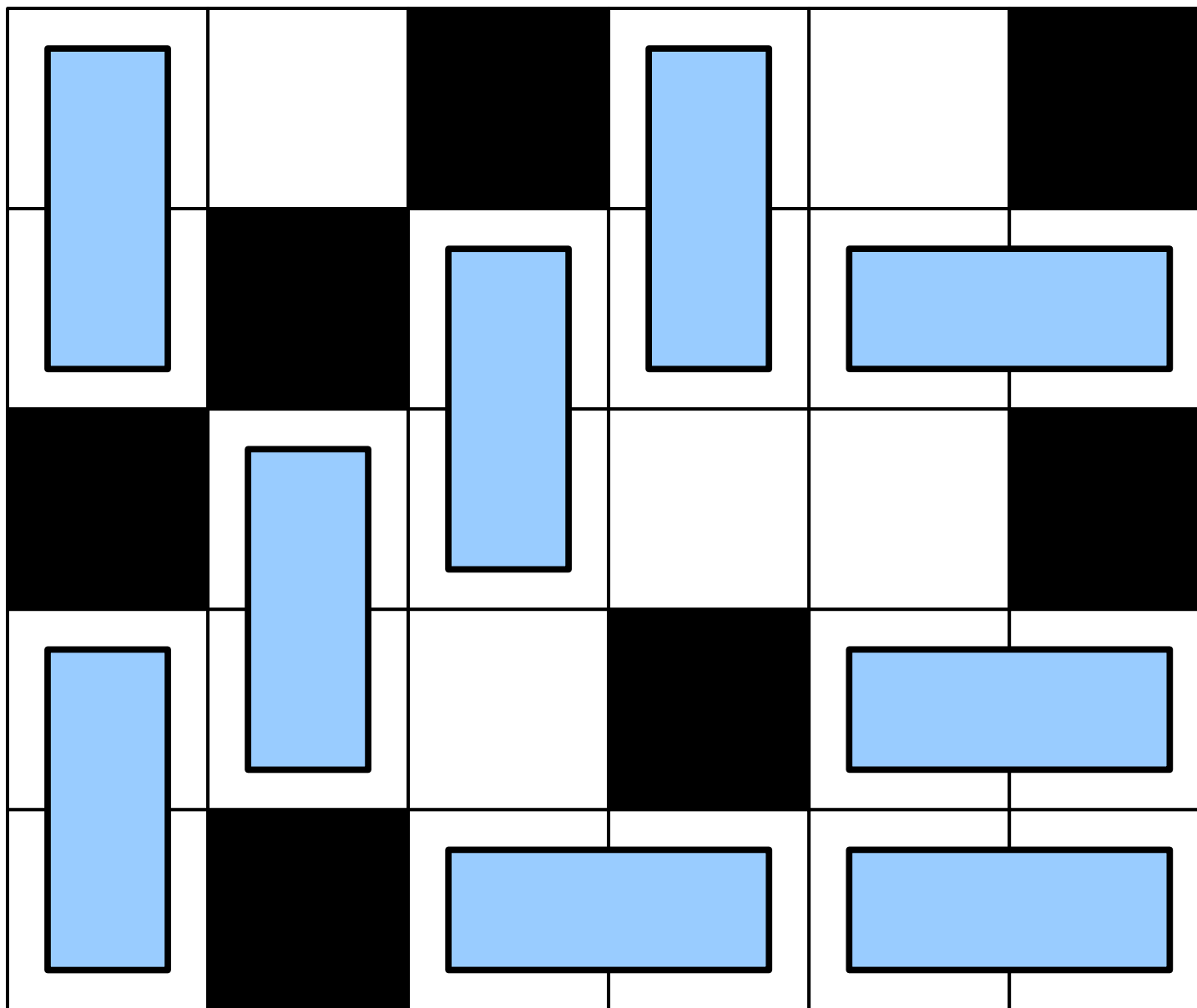
Maximum Matching

- Jack Edmonds' paper “Paths, Trees, and Flowers” gives a polynomial-time algorithm for finding maximum matchings.
 - He's the guy from last time with the quote about “better than decidable.”
- Using this fact, what other problems can we solve?

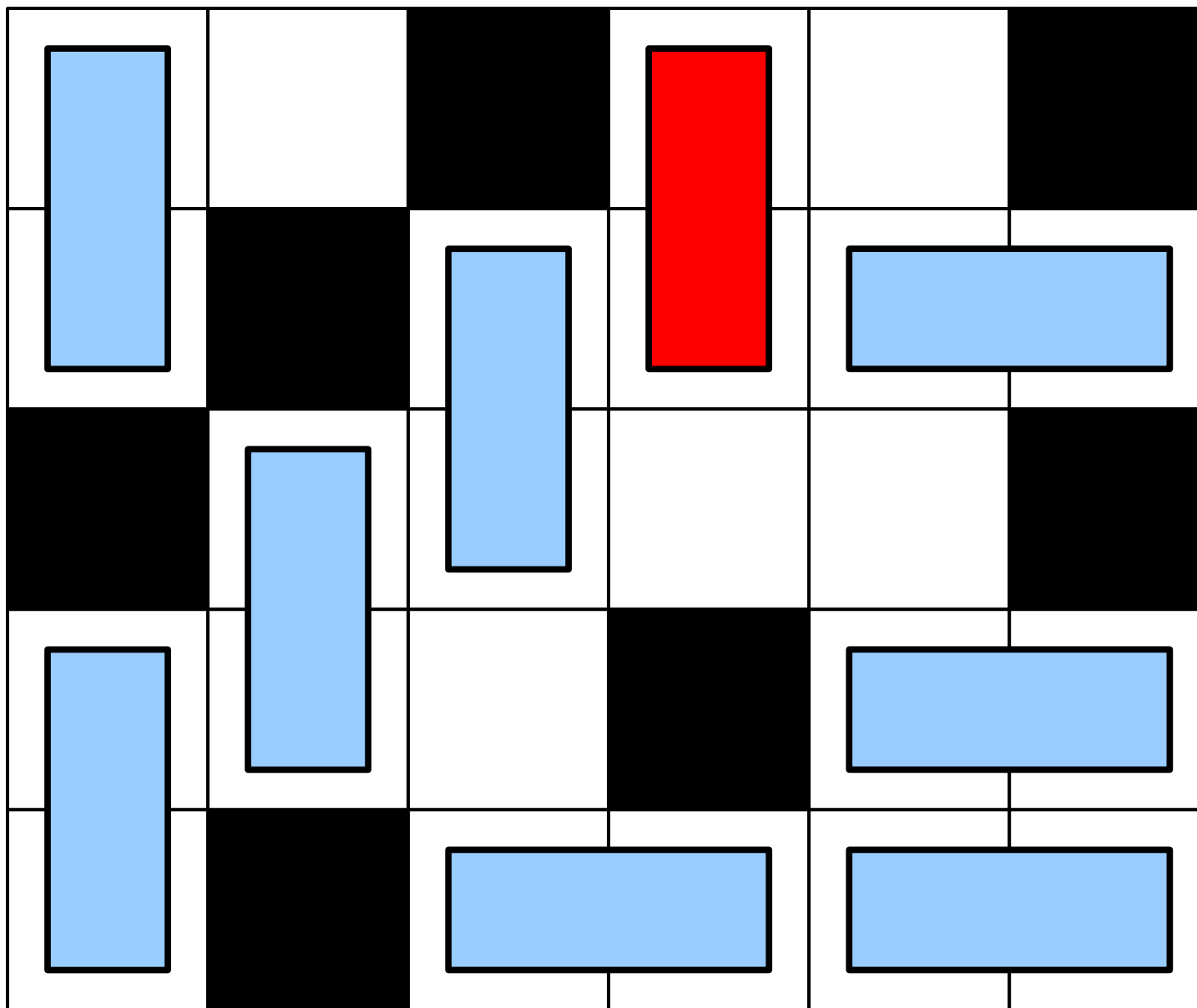
Domino Tiling



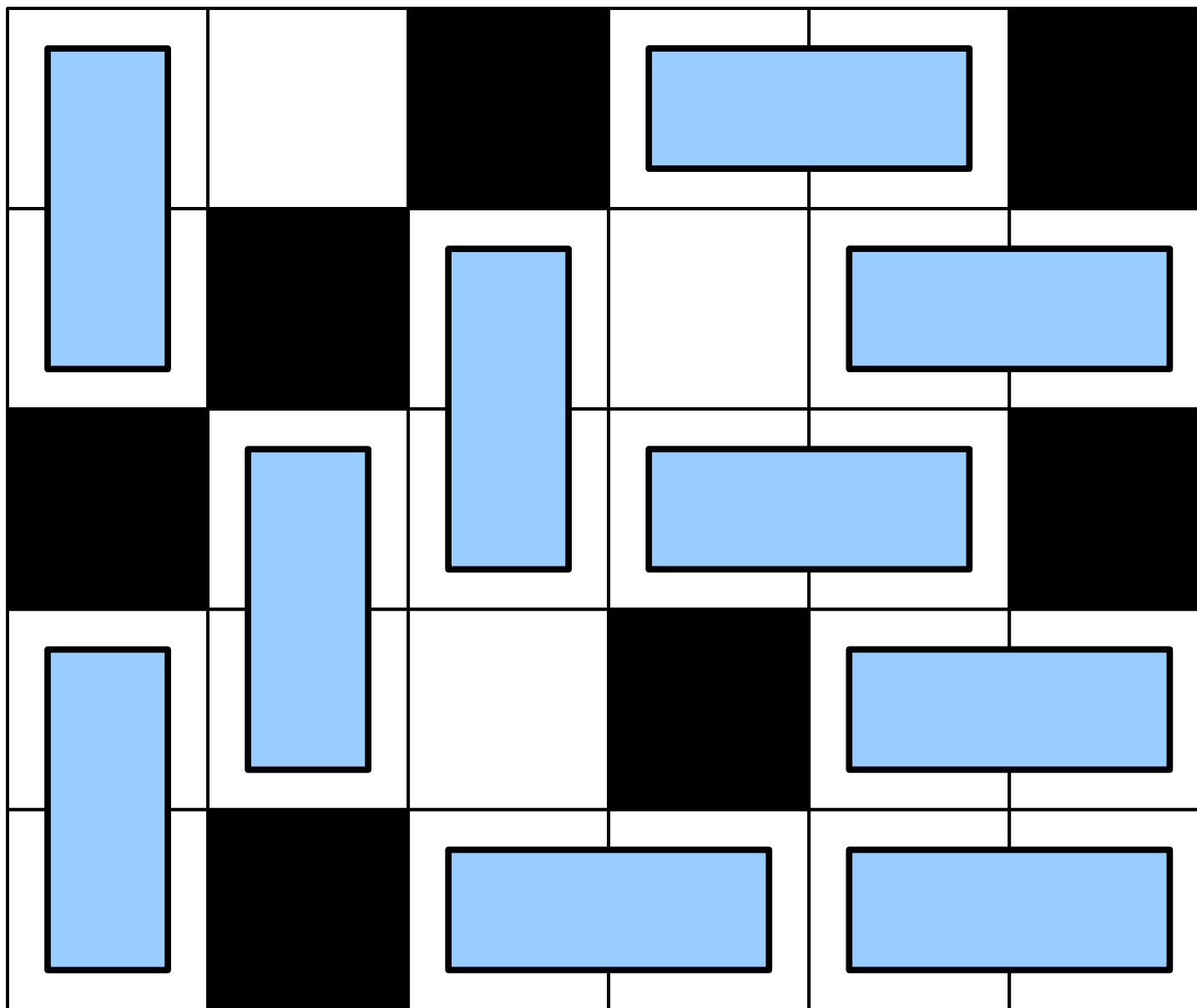
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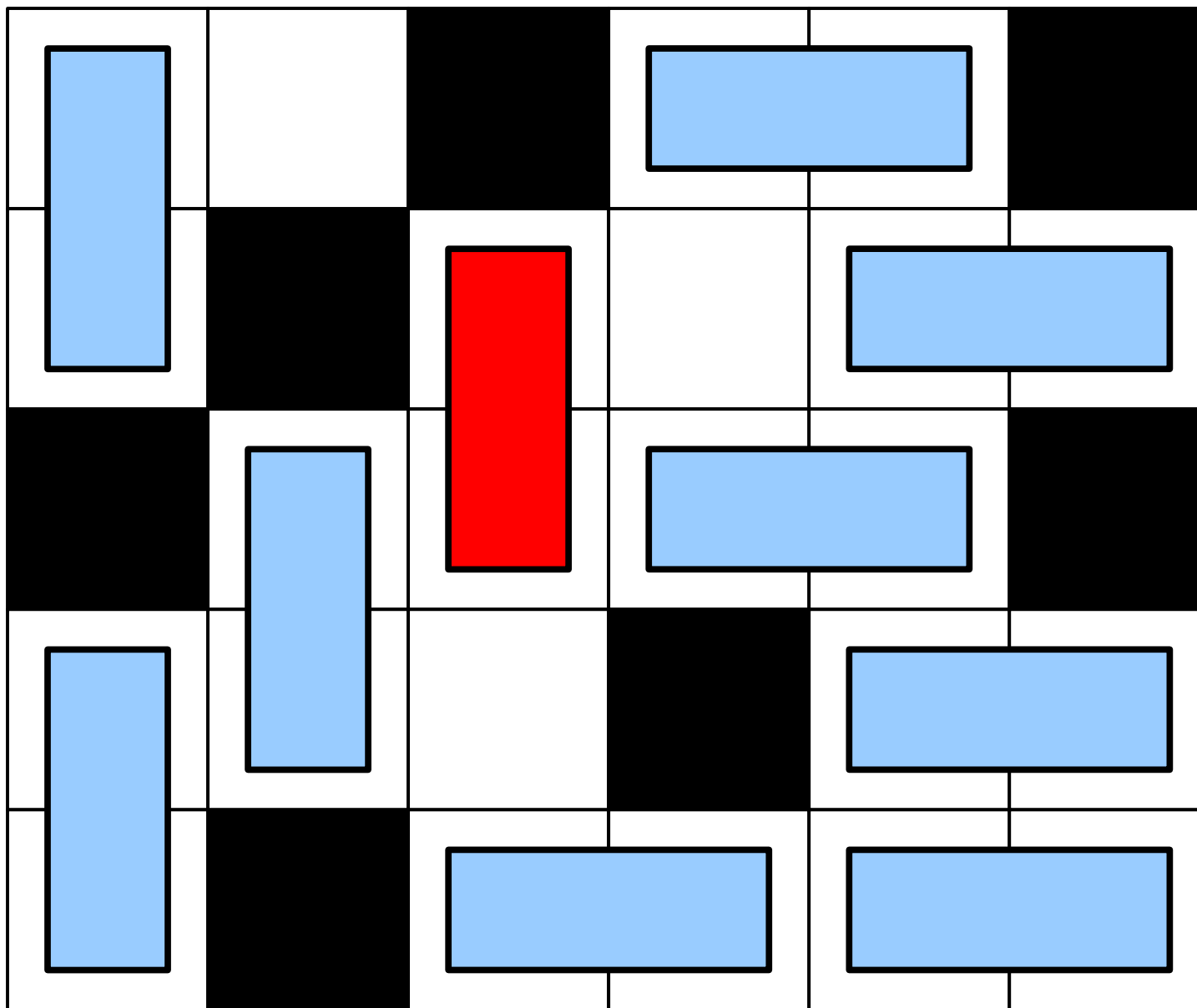
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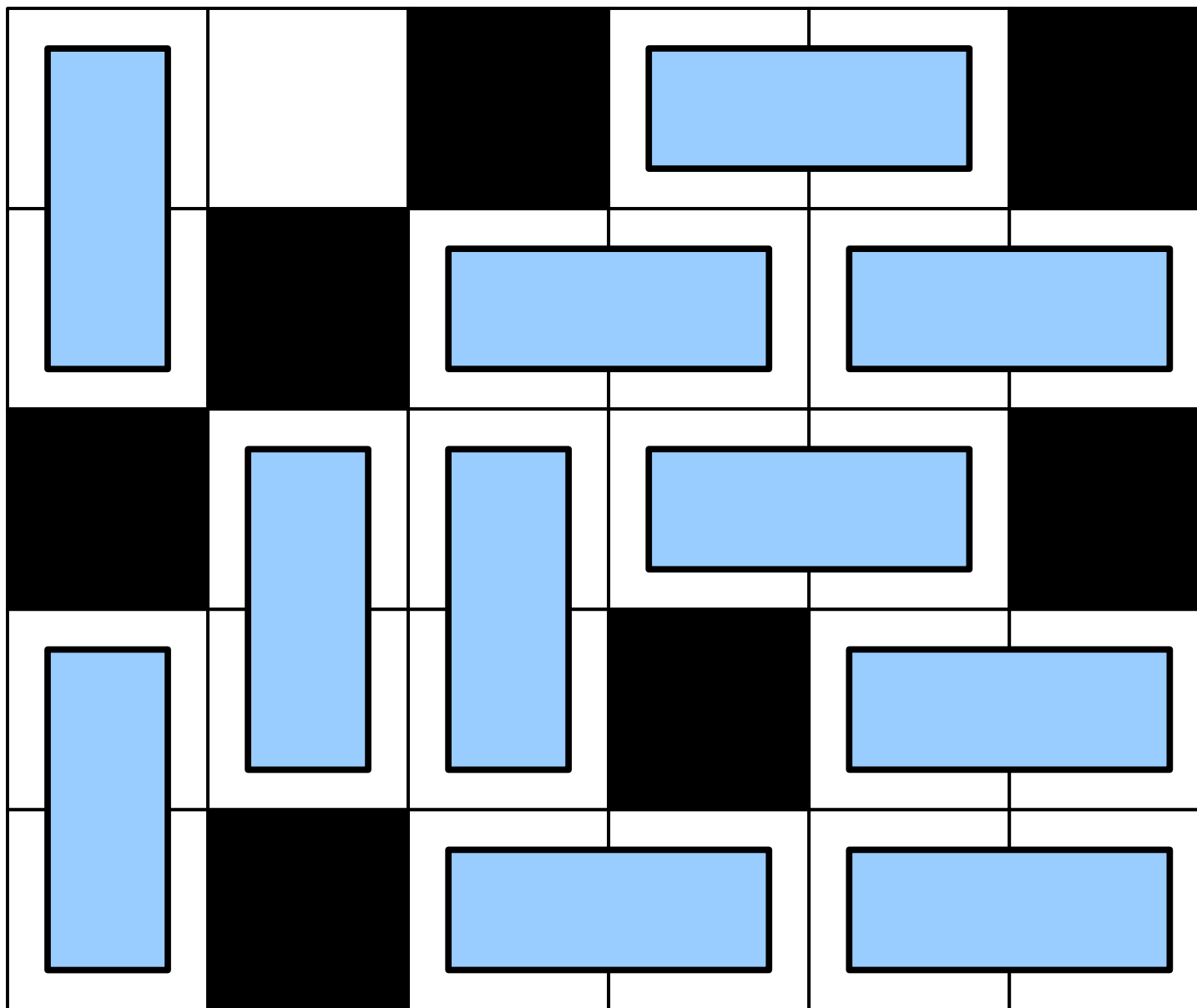
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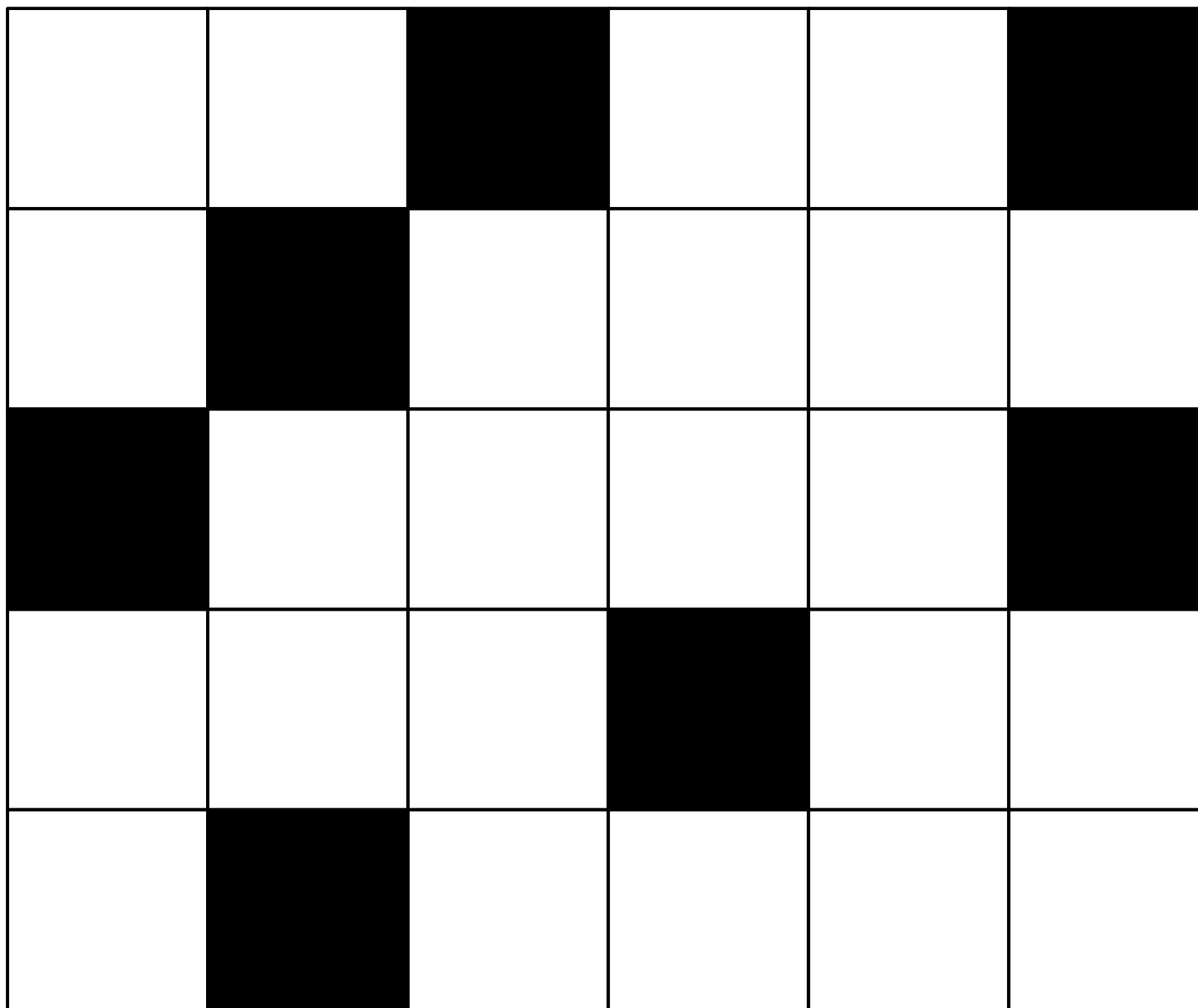
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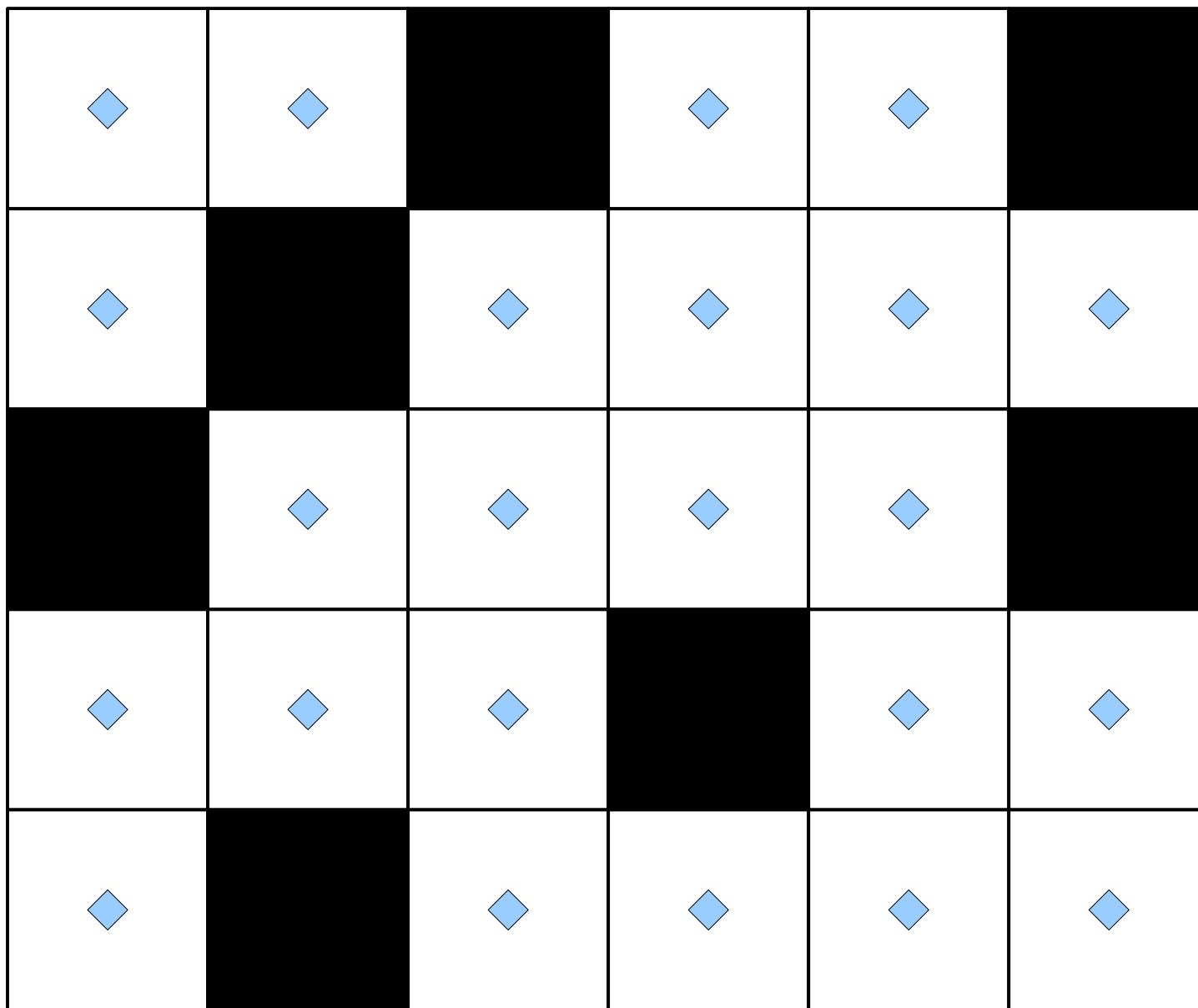
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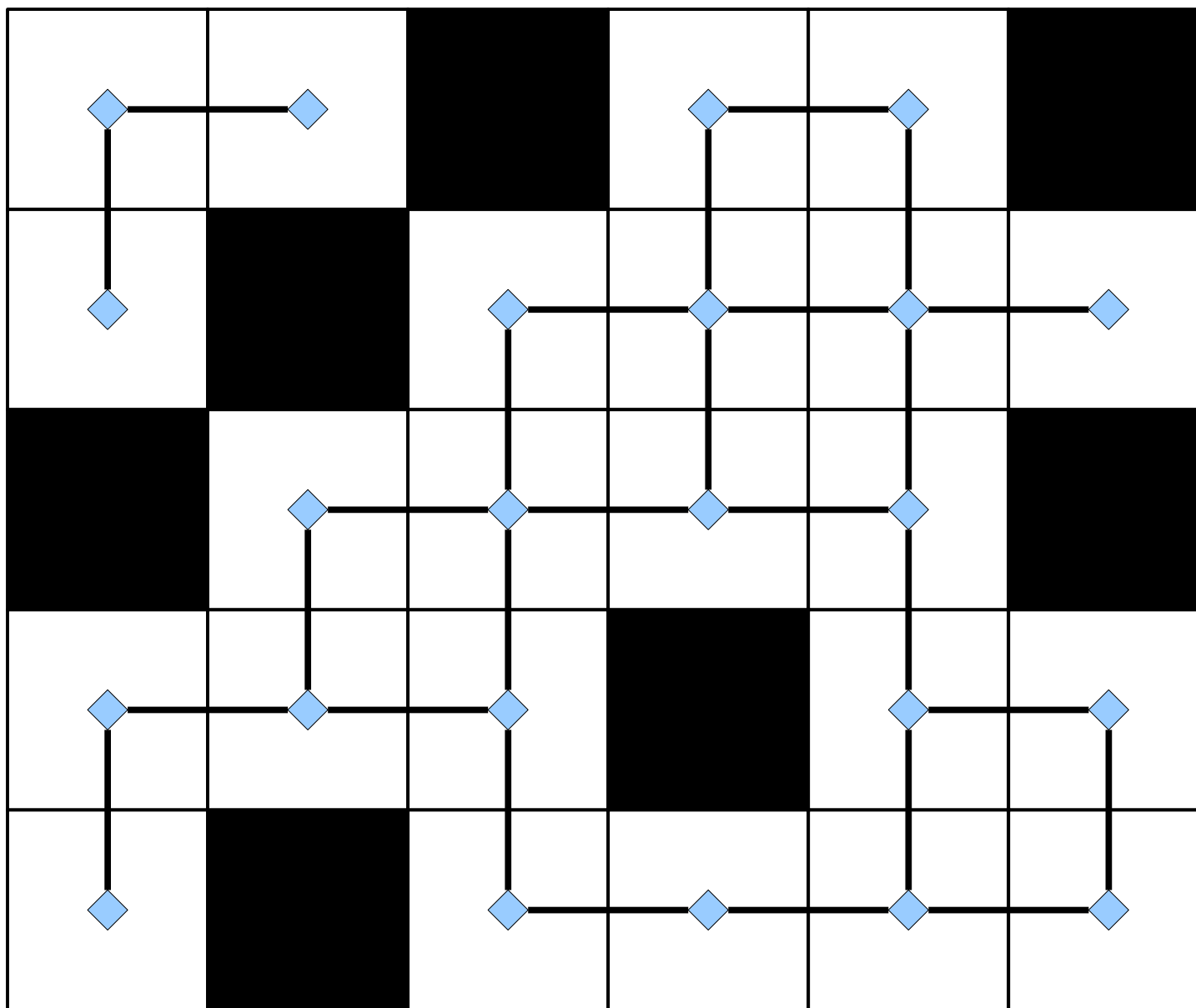
Solving Domino Tiling



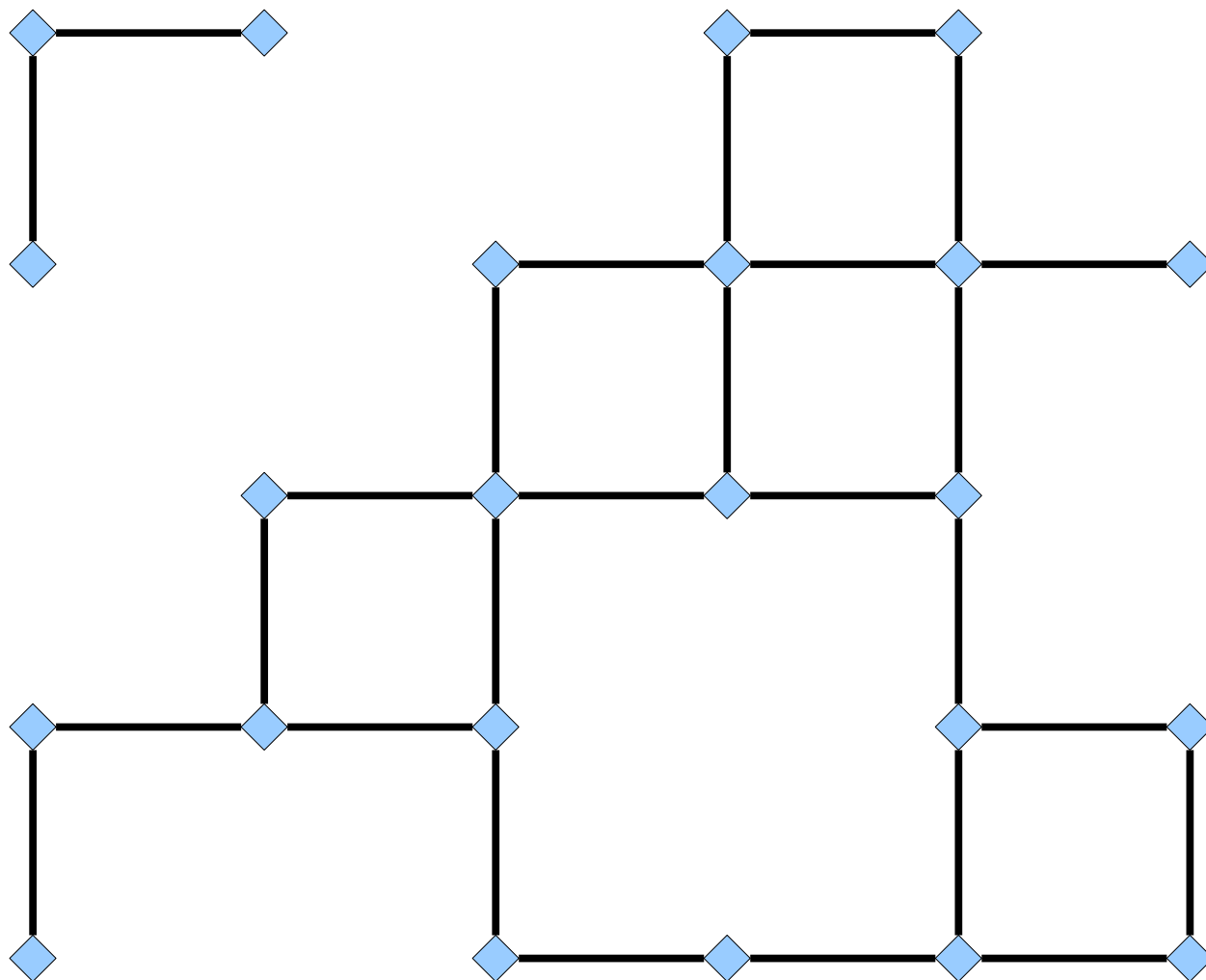
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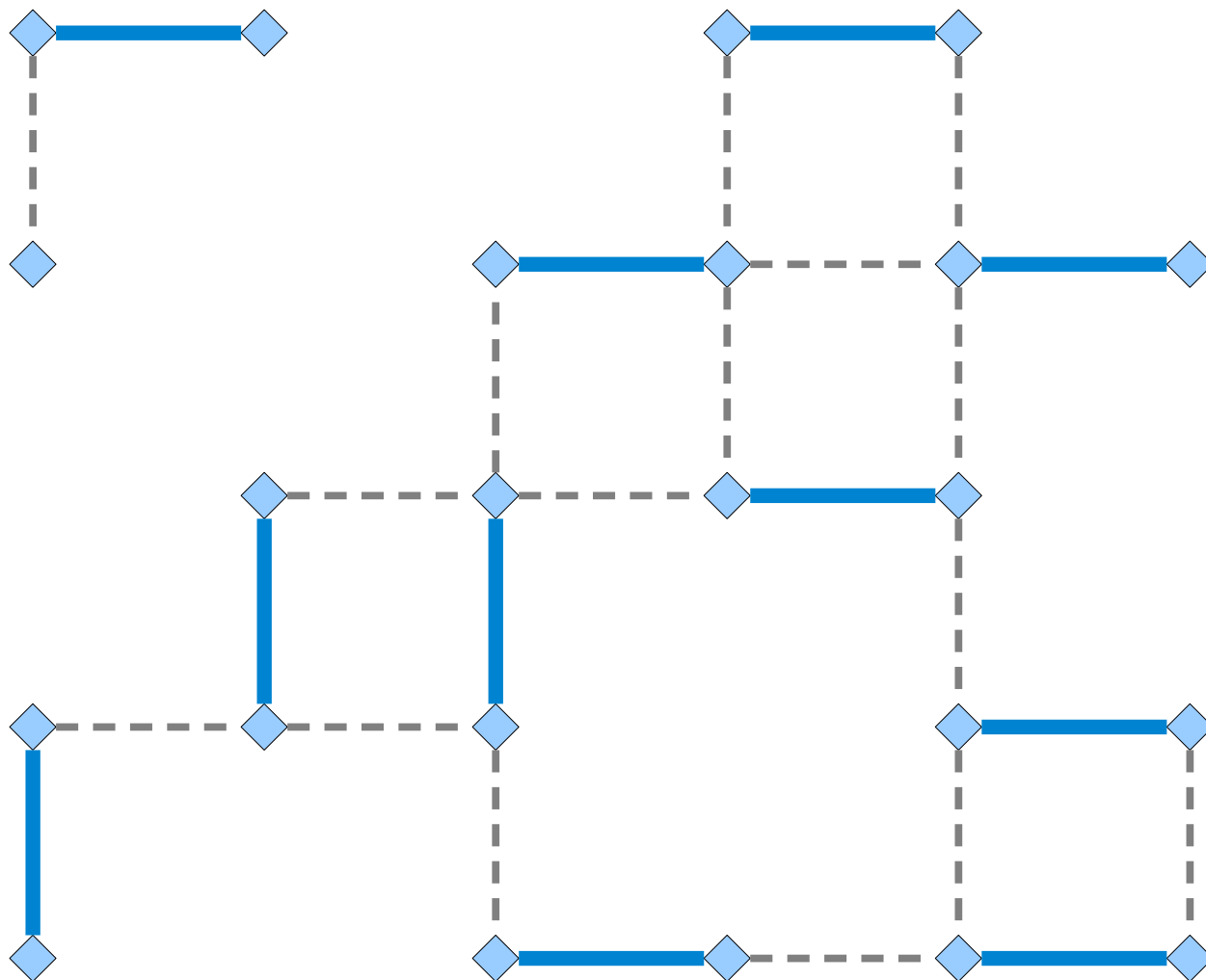
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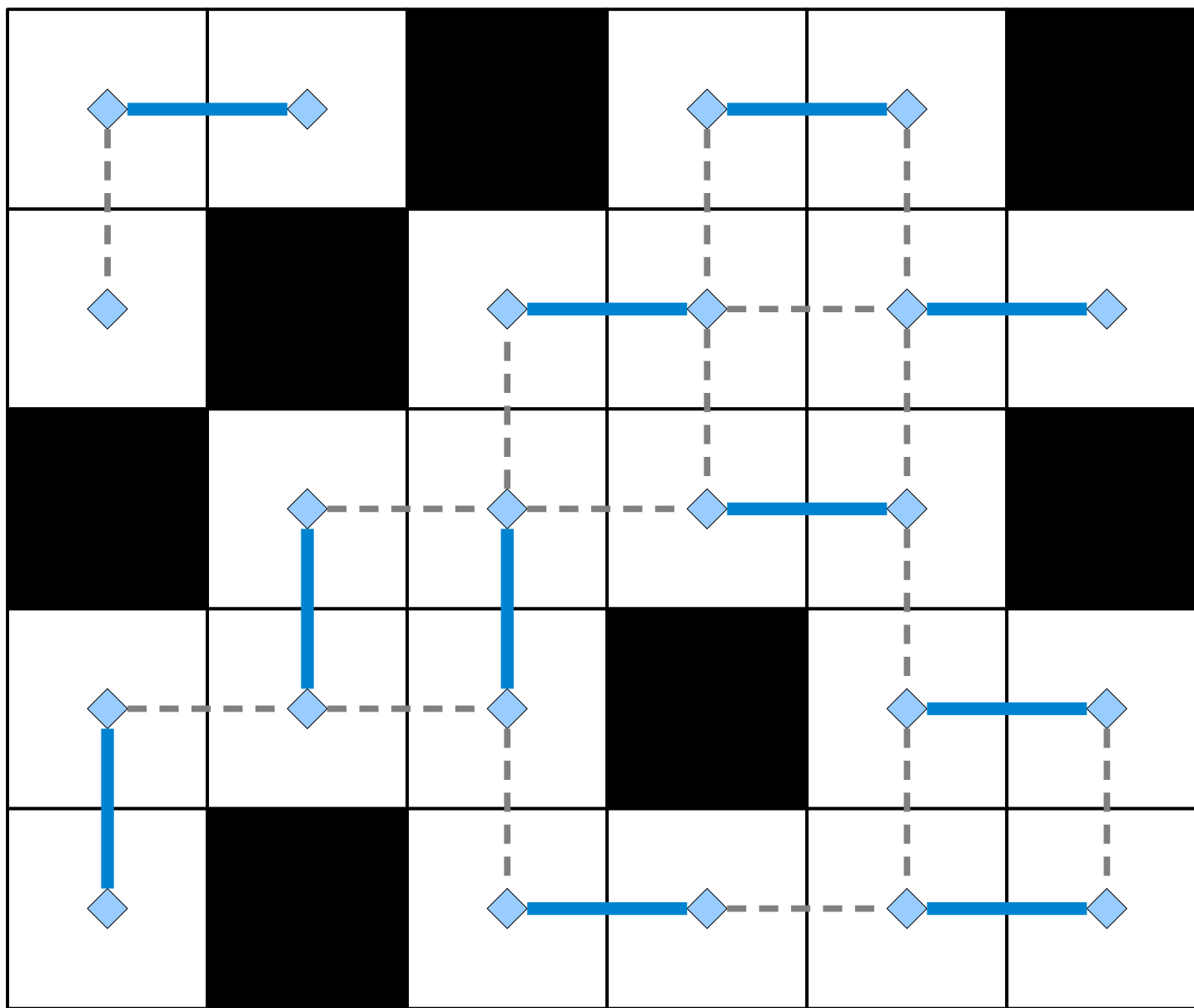
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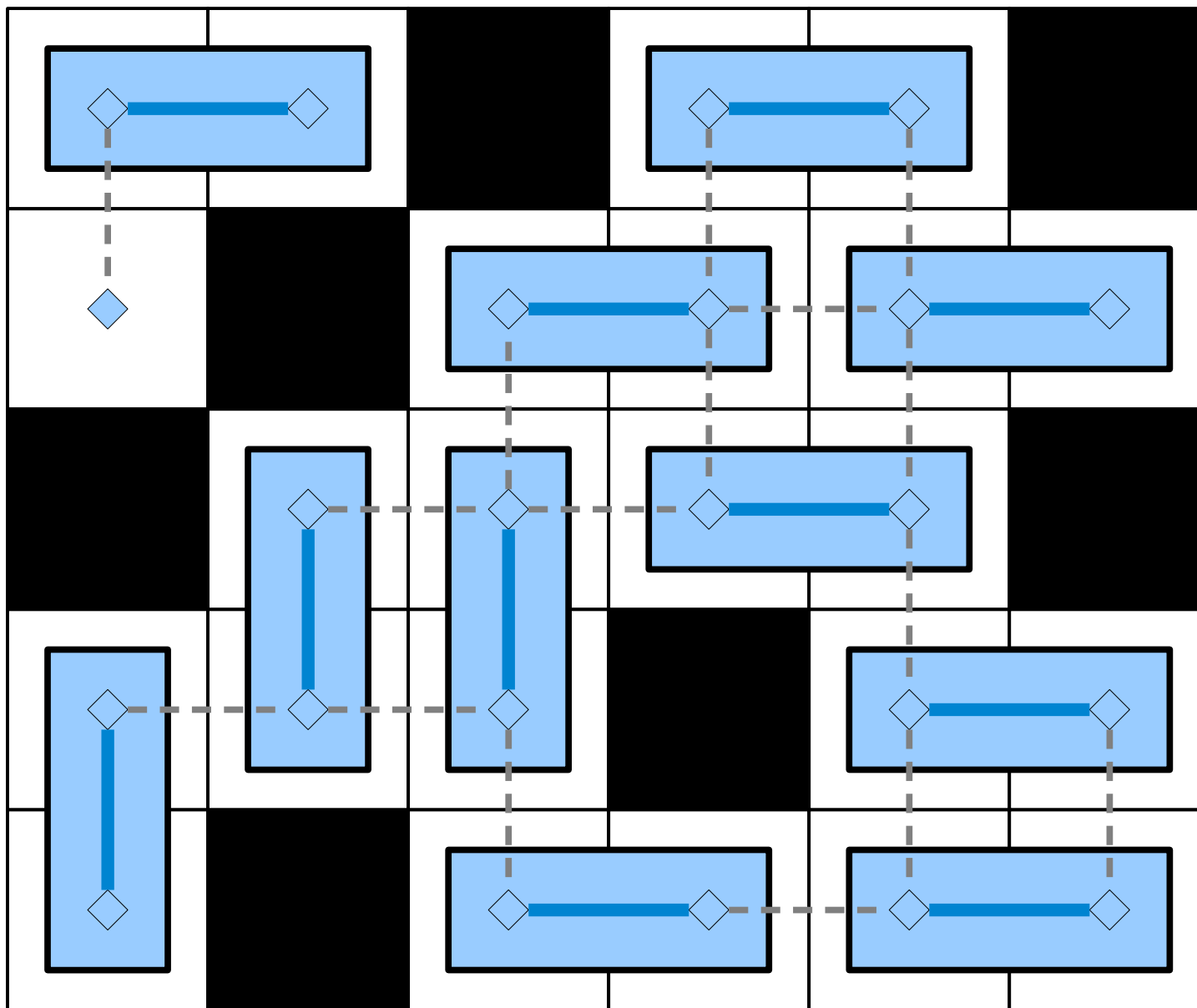
Solving Domino Tiling



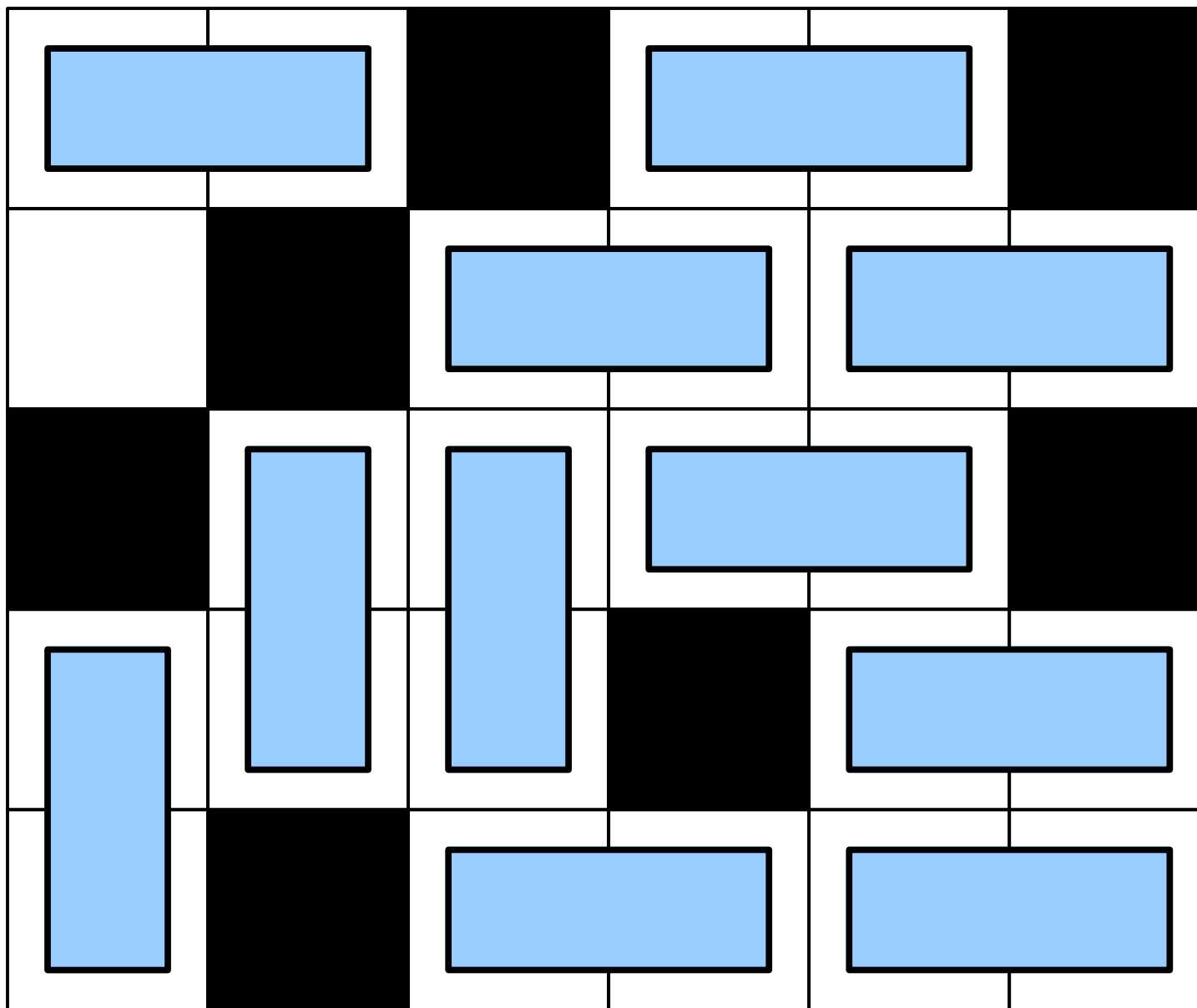
Solving Domino Tiling



Solving Domino Tiling



Solving Domino Tiling



```
bool canPlaceDominoes(Grid G, int k) {  
    return hasMatching(gridToGraph(G), k);  
}
```

Which of the following is the most reasonable conclusion to draw, given the existence of the above function?

- A. Solving domino tiling on a 2D grid can't be "harder" than solving maximum matching.
- B. Solving maximum matching can't be "harder" than solving domino tiling on a 2D grid.
- C. Both A and B.

Answer at

<https://cs103.stanford.edu/pollev>

Intuition:

Tiling a grid with dominoes can't be “harder” than solving maximum matching, because if we can solve maximum matching efficiently, we can solve domino tiling efficiently.

Another Example

Satisfiability

- A propositional logic formula φ is called **satisfiable** if there is some assignment to its variables that makes it evaluate to true.
- Which of the following formulas are satisfiable?

$$p \wedge q$$

$$p \wedge \neg p$$

$$p \rightarrow (q \wedge \neg q)$$

- An assignment of true and false to the variables of φ that makes it evaluate to true is called a **satisfying assignment**.

SAT

- The ***boolean satisfiability problem*** (***SAT***) is the following:

Given a propositional logic formula φ , is φ satisfiable?

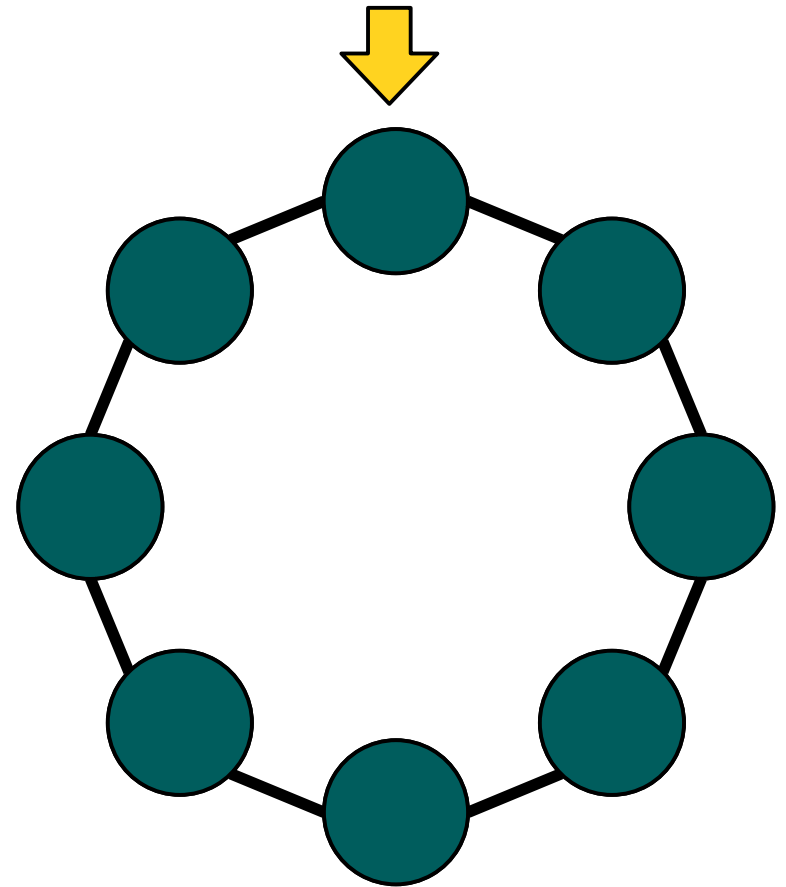
- Formally:

$SAT = \{ \langle \varphi \rangle \mid \varphi \text{ is a satisfiable PL formula} \}$

- Finding good algorithms for SAT is an active area of research for reasons we'll discuss later today.
- We have some pretty decent algorithms for solving SAT reasonably quickly most of the time.
- Given this, what other problems can we solve?

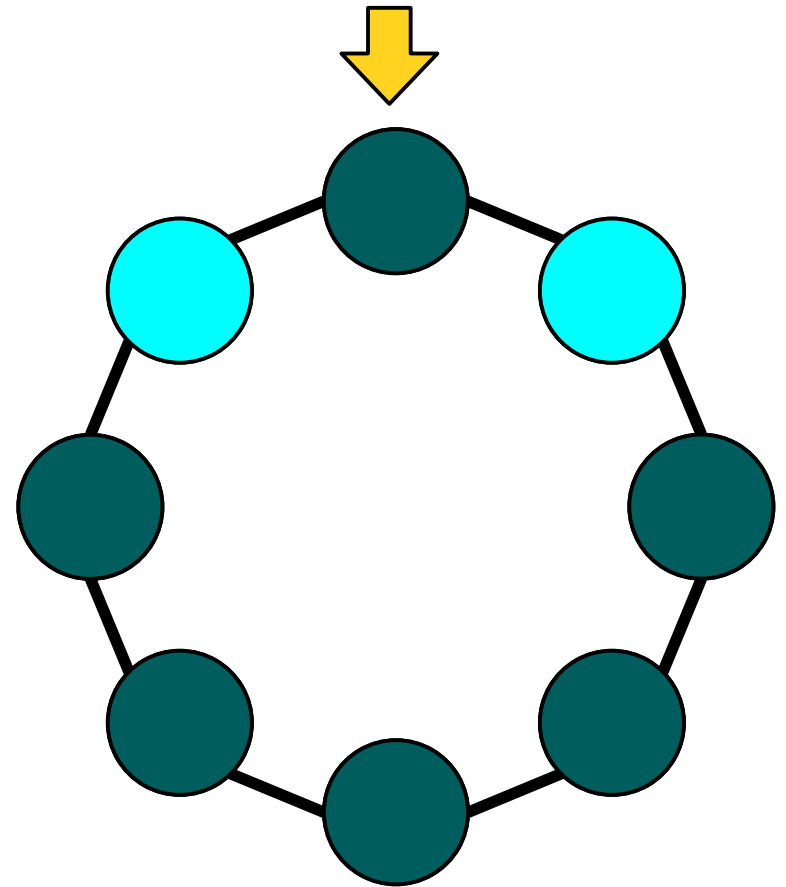
Lights Out

- You're given a ring of pushbuttons. Each pushbutton has a light that is either ON or OFF.
- If you push a button, it toggles the state of the two adjacent lights in the ring. (Lights that are ON turn OFF and vice-versa.)



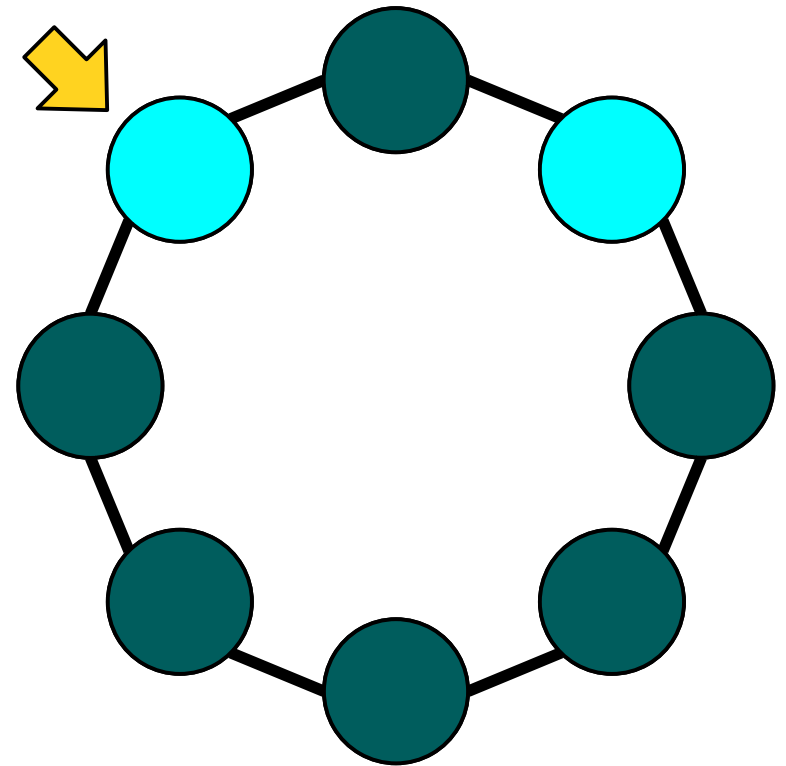
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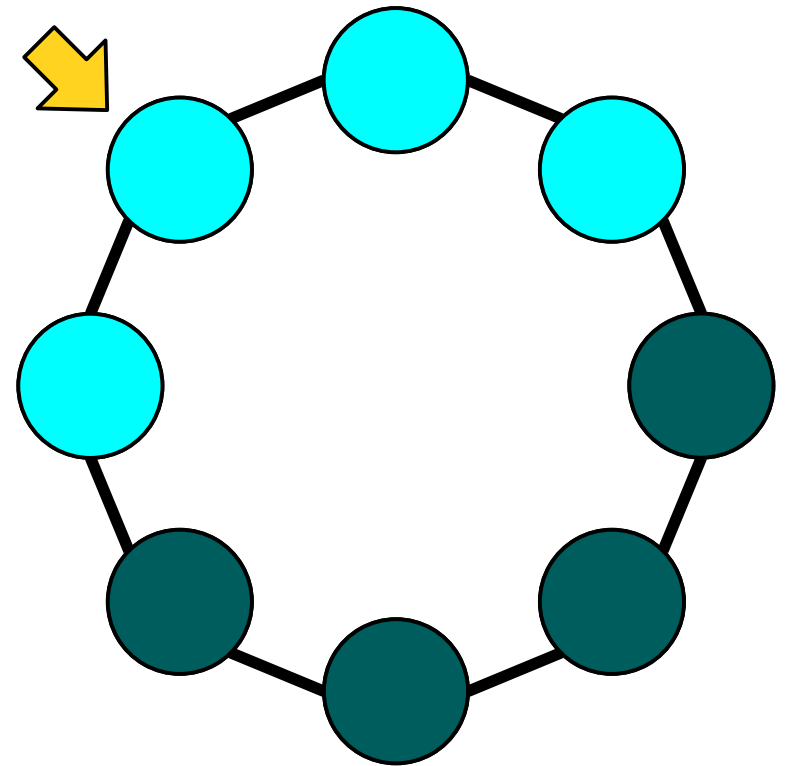
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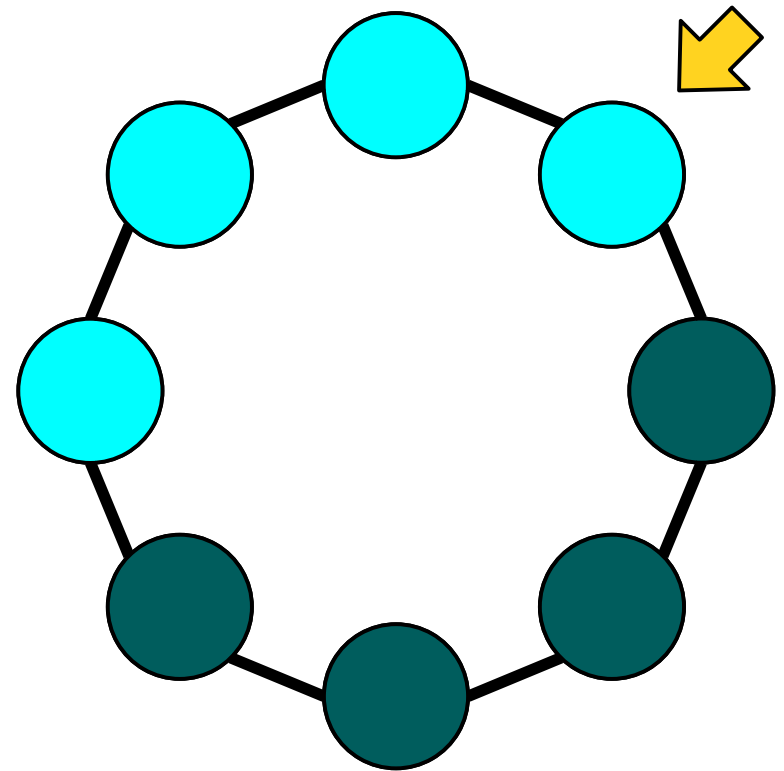
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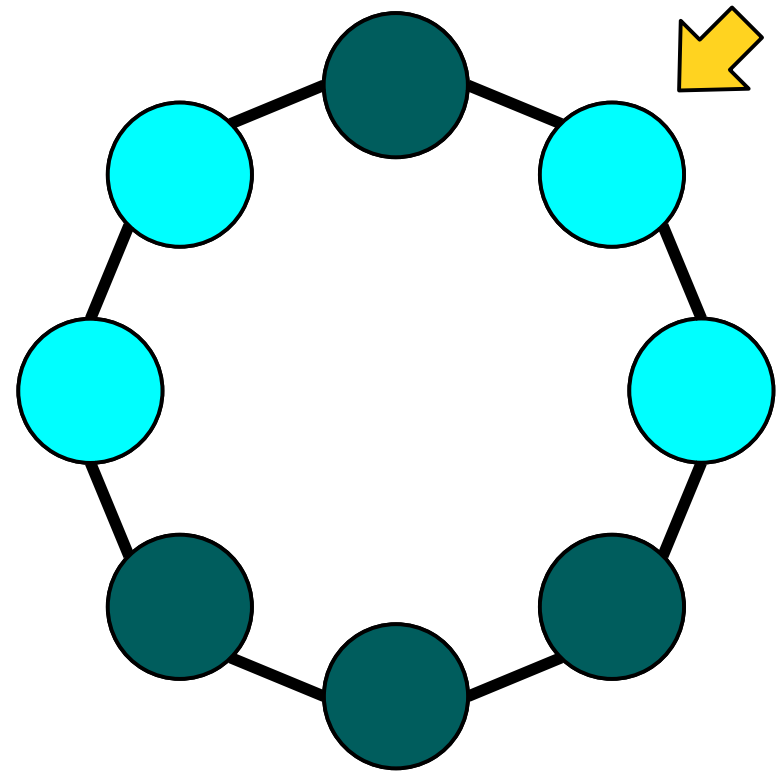
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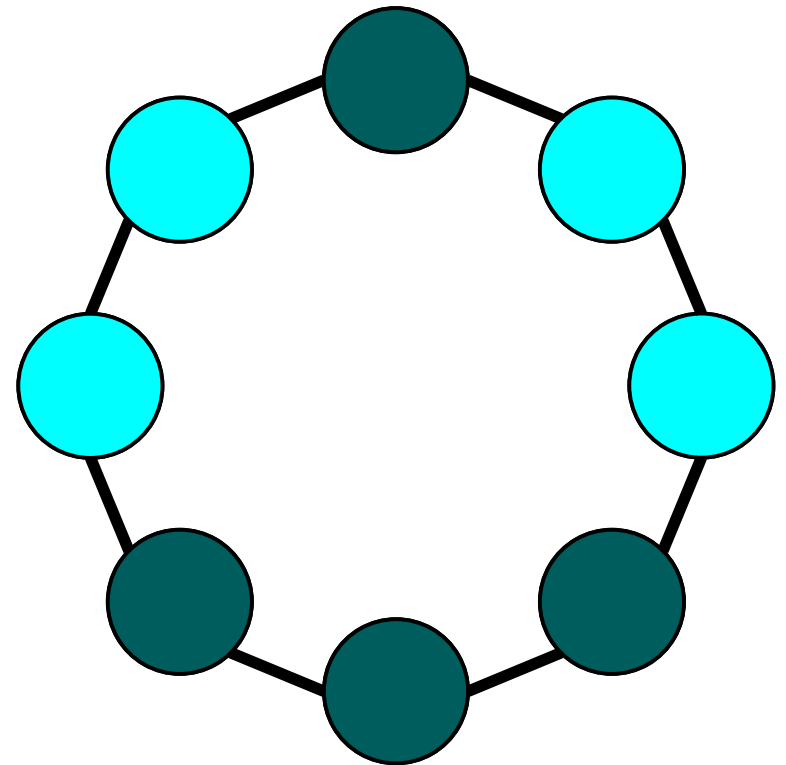
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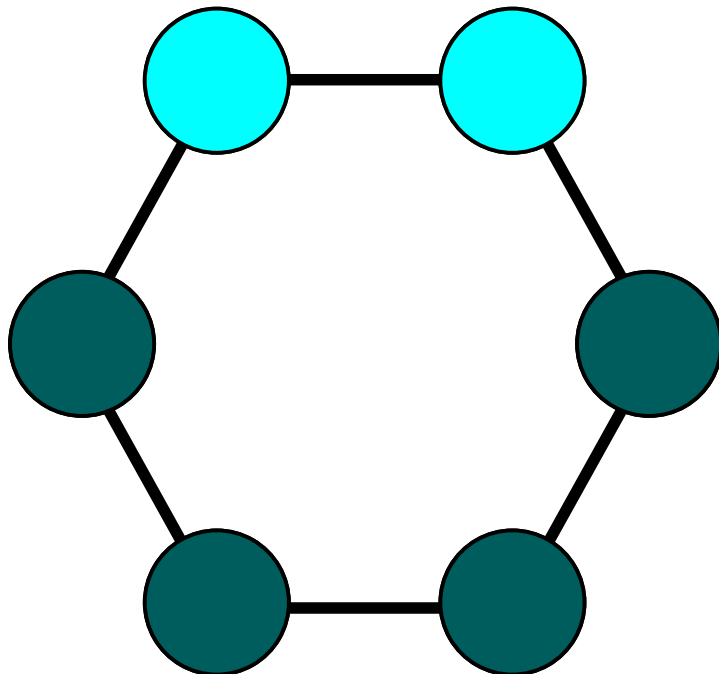
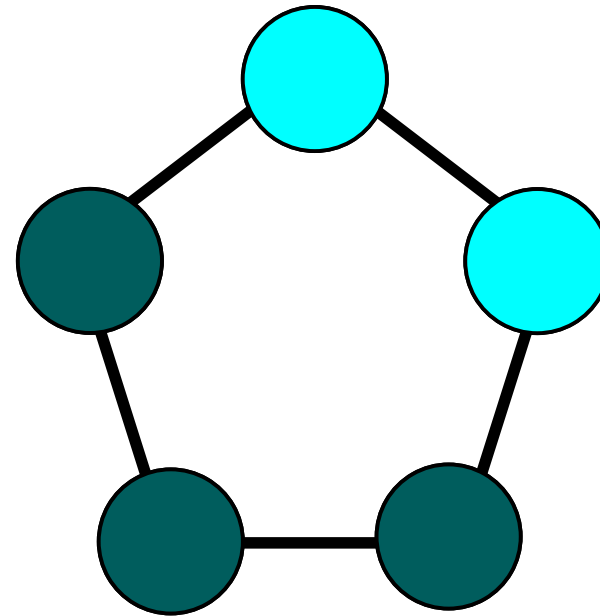
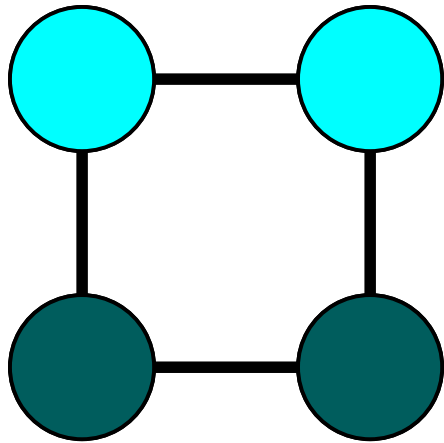
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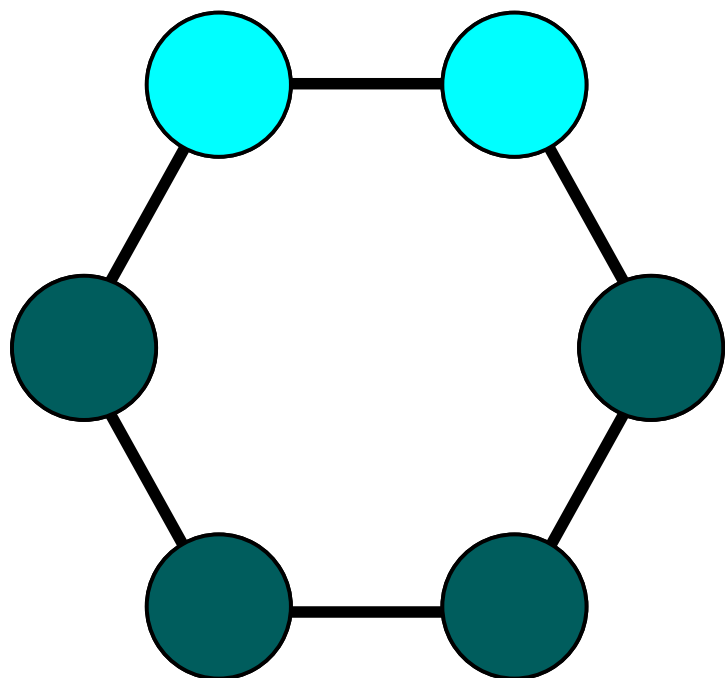
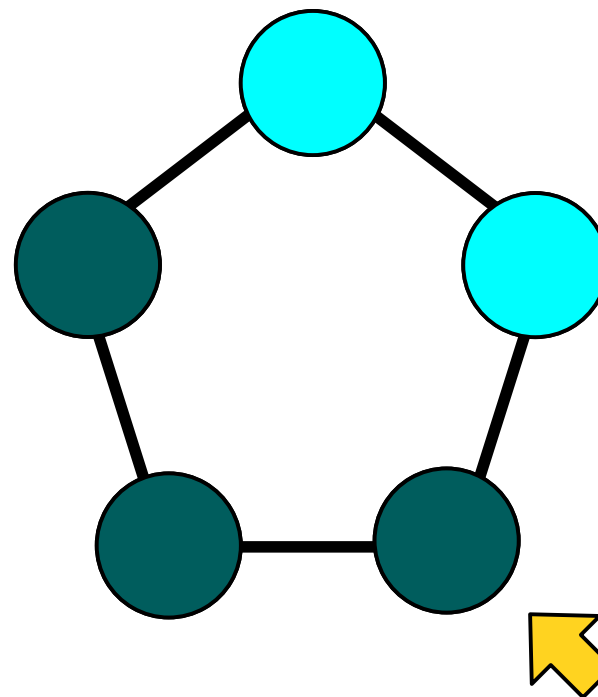
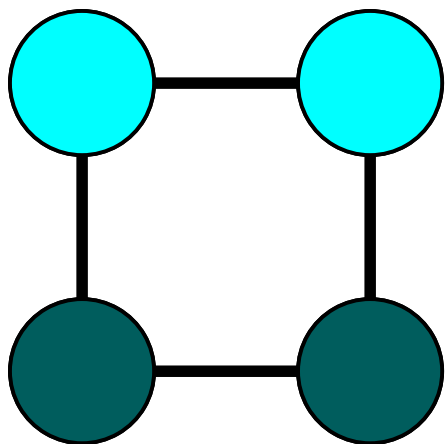
- You're given a ring of pushbuttons. Each pushbutton has a light that is either ON or OFF.
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- **Question:** Given an initial configuration of lights, can you turn all the lights off?

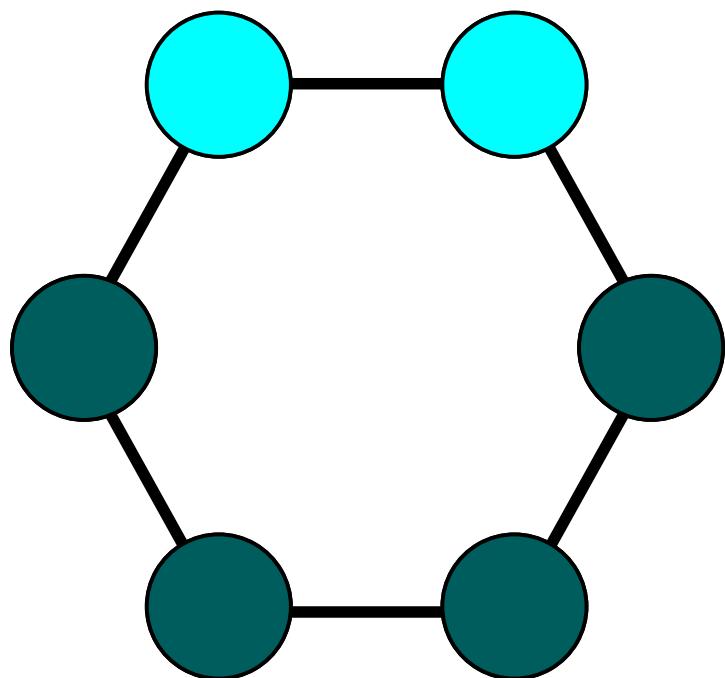
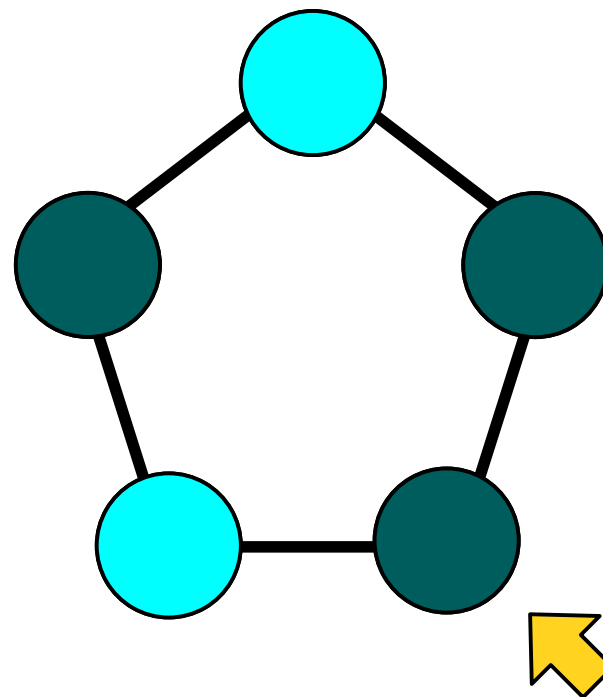
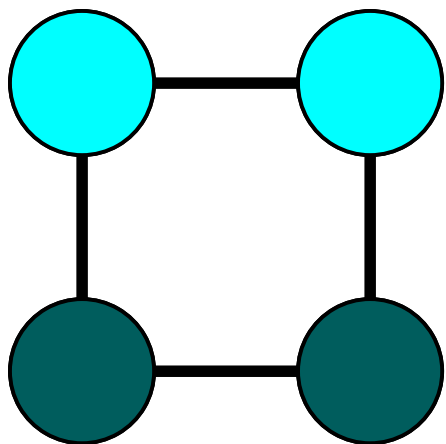


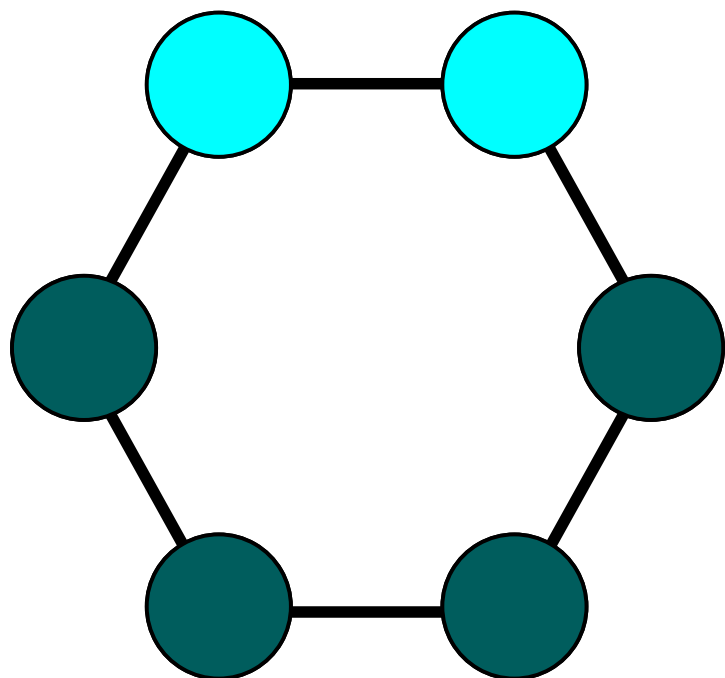
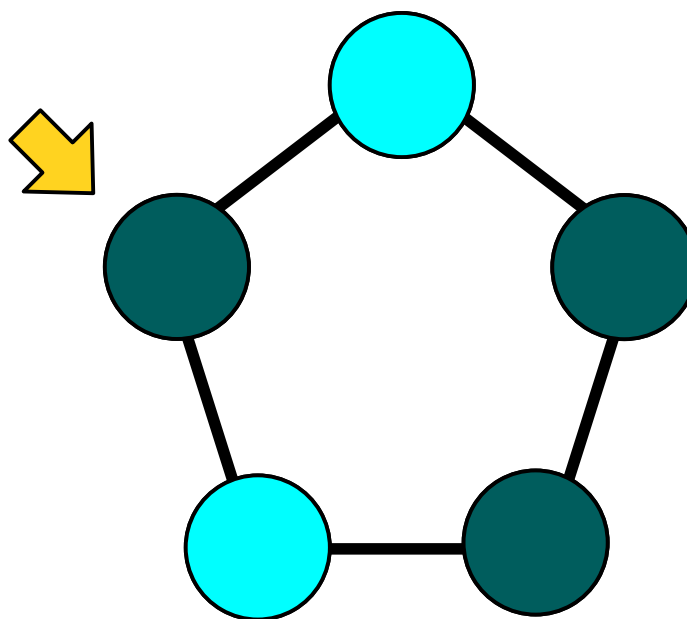
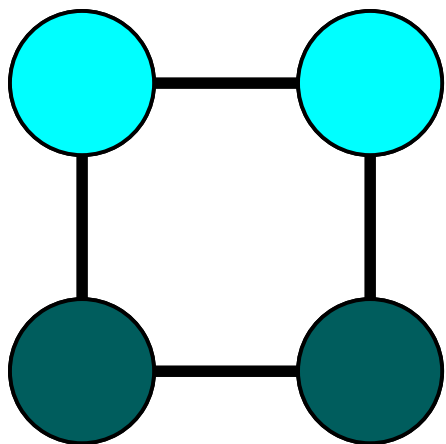


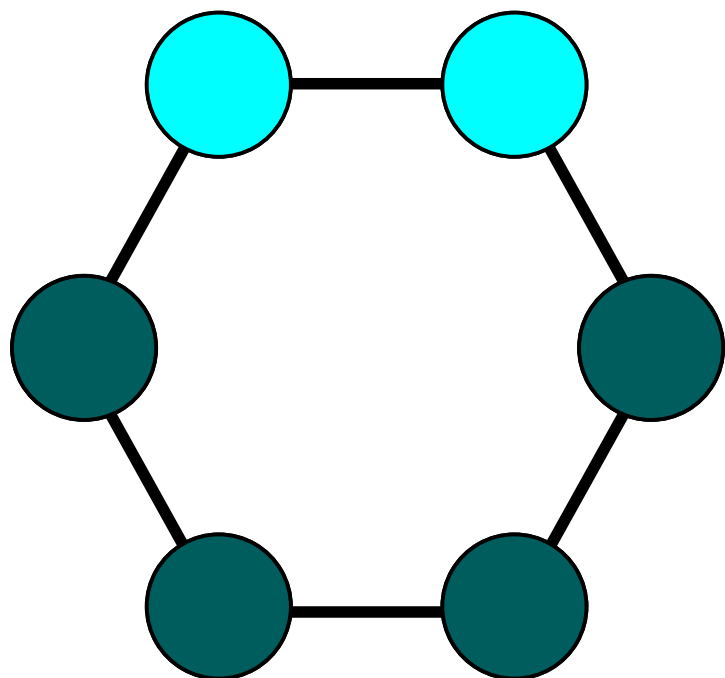
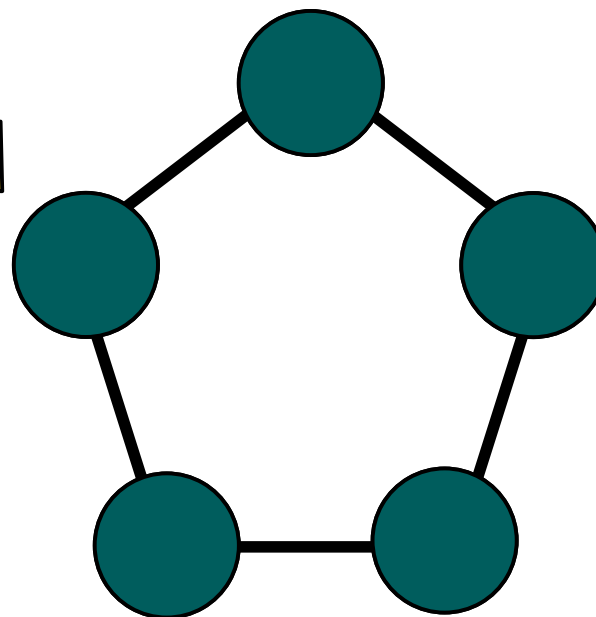
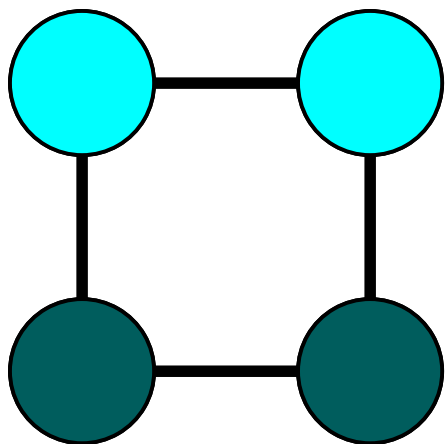
In which of these rings can you
turn off all the lights?

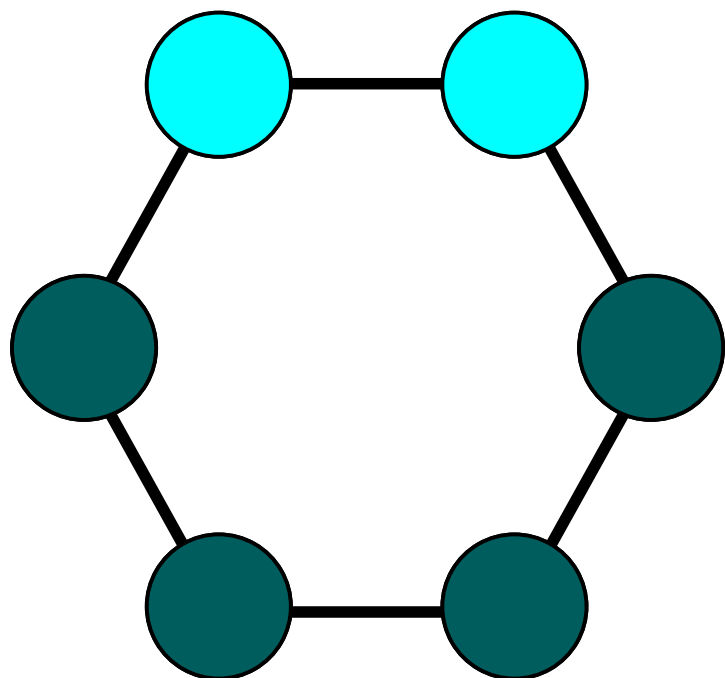
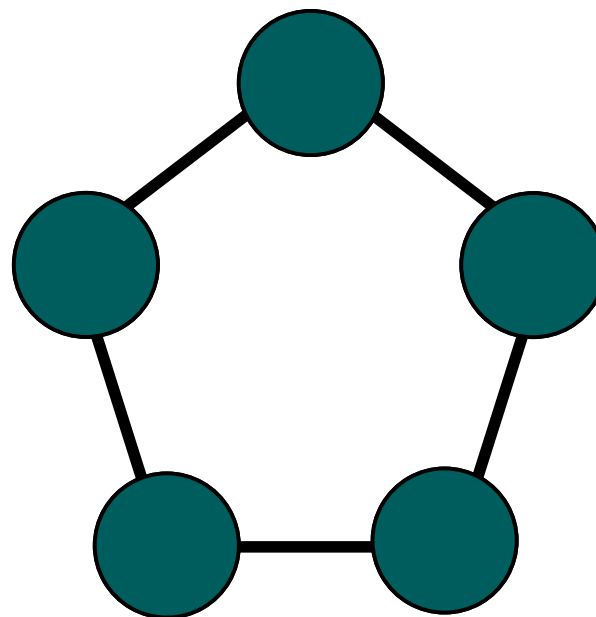
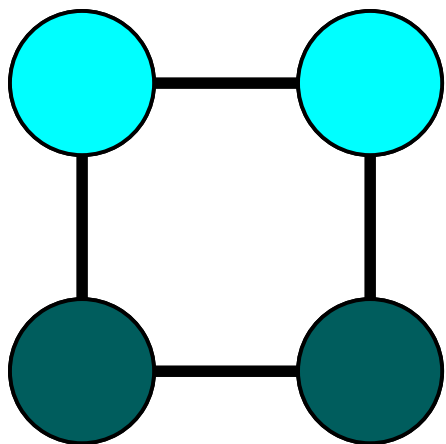
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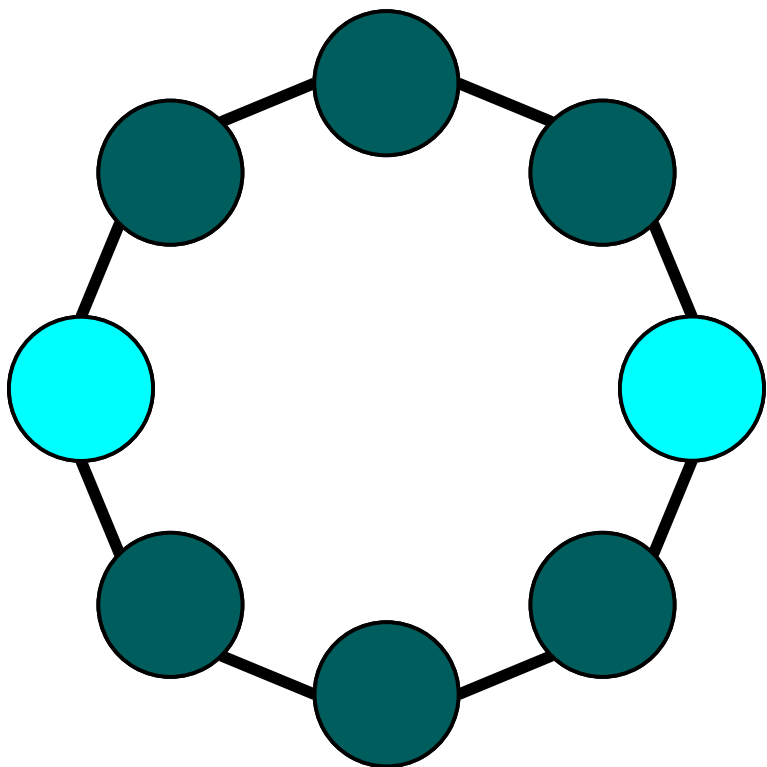


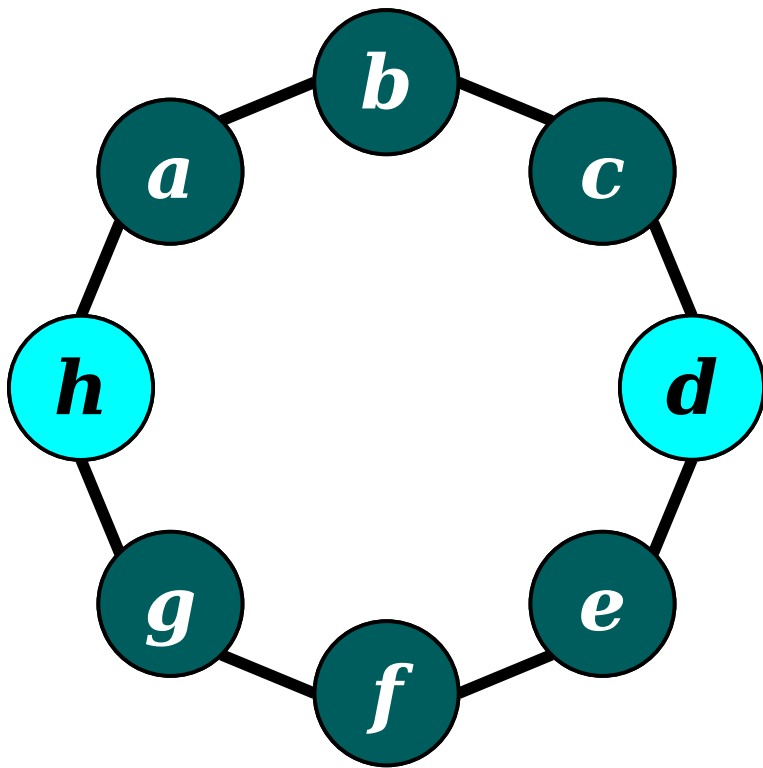






Solving Lights-Out With a SAT Solver





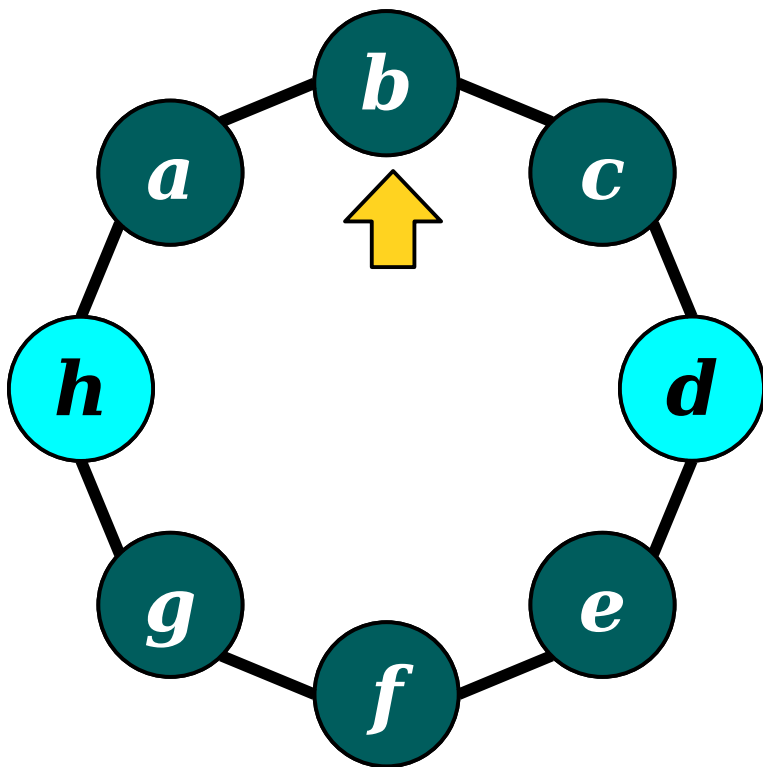
Observation 1: We never need to press the same button twice.

Observation 2: Button press order doesn't matter.

Observation 3: Our propositional formula will have one variable per button, indicating whether we press it.

Observation 4: A light that is initially off stays off when an even number of adjacent lights are pressed.

Observation 5: A light that is initially on ends off when an odd number of adjacent lights are pressed.



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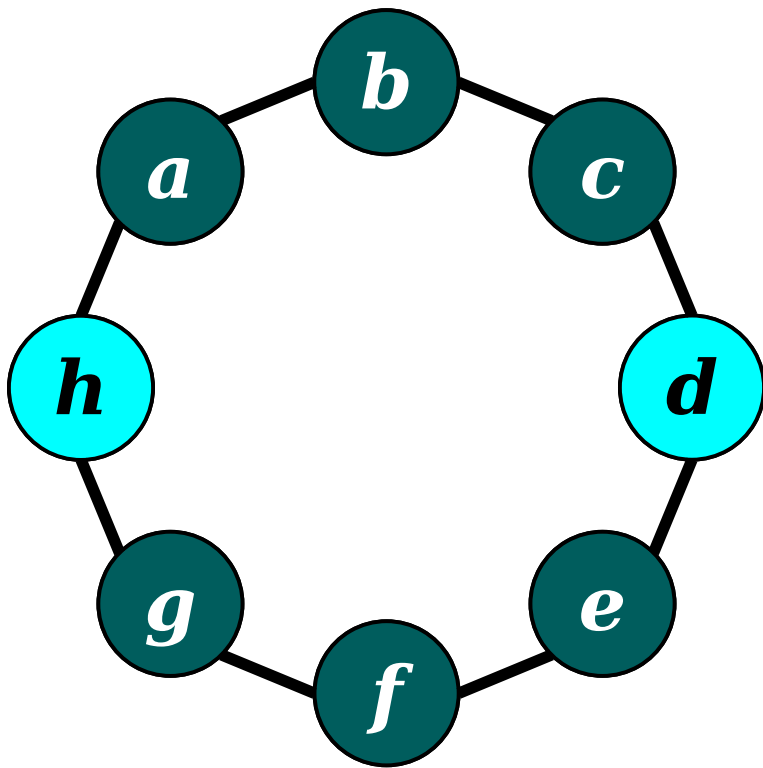
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Write a formula in propositional logic that says “an even number of the variables *a* and *c* are true.”

Answer at

<https://cs103.stanford.edu/pollev>


$$\begin{aligned}(h \leftrightarrow b) \wedge \\ (a \leftrightarrow c) \wedge \\ (b \leftrightarrow d) \wedge \\ \neg(c \leftrightarrow e) \wedge \\ (d \leftrightarrow f) \wedge \\ (e \leftrightarrow g) \wedge \\ (f \leftrightarrow h) \wedge \\ \neg(a \leftrightarrow g)\end{aligned}$$

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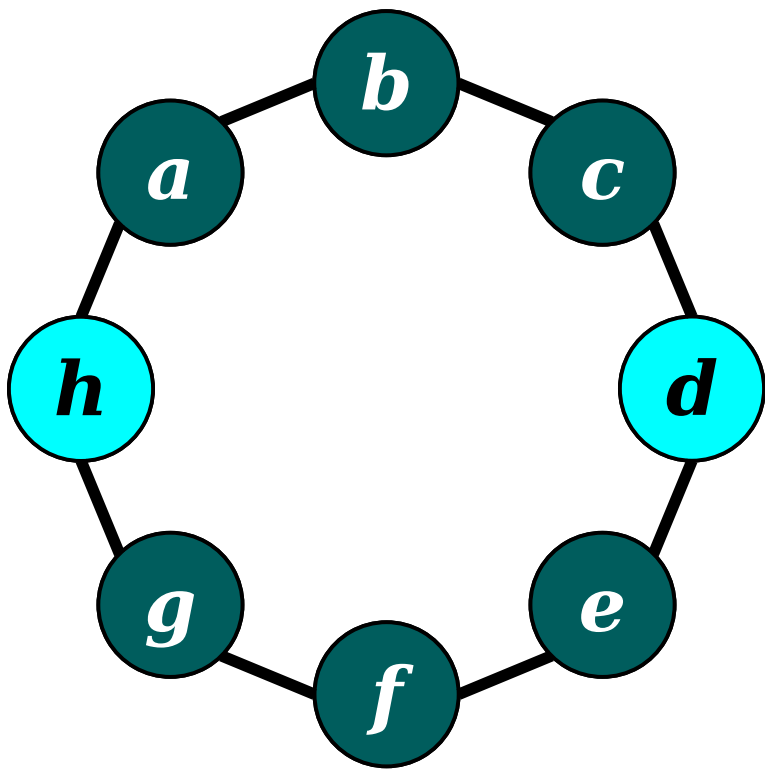
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 $\neg(a \leftrightarrow g)$

Make a and c true.
Make b, d, e, f, g , and h false.

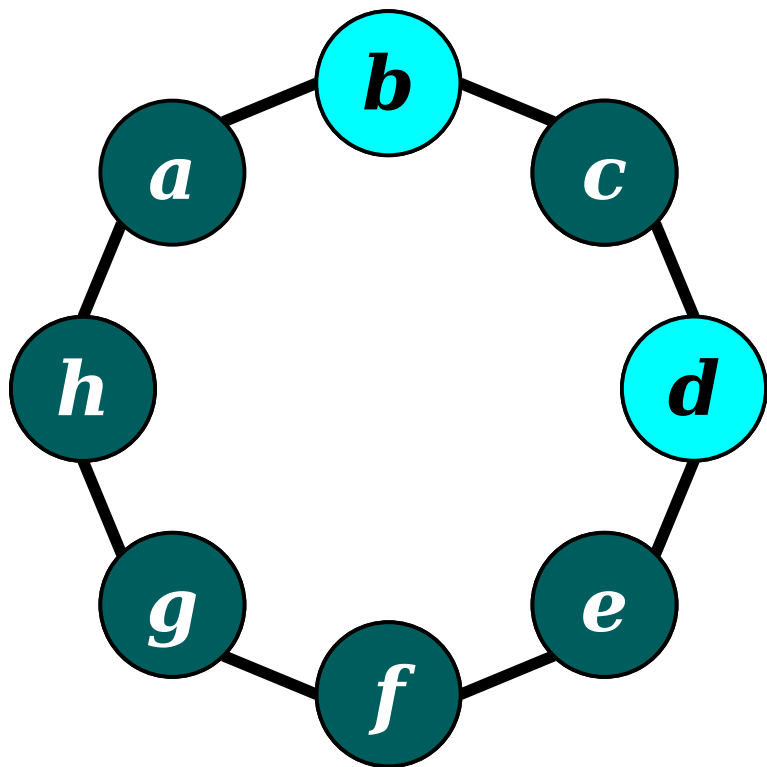
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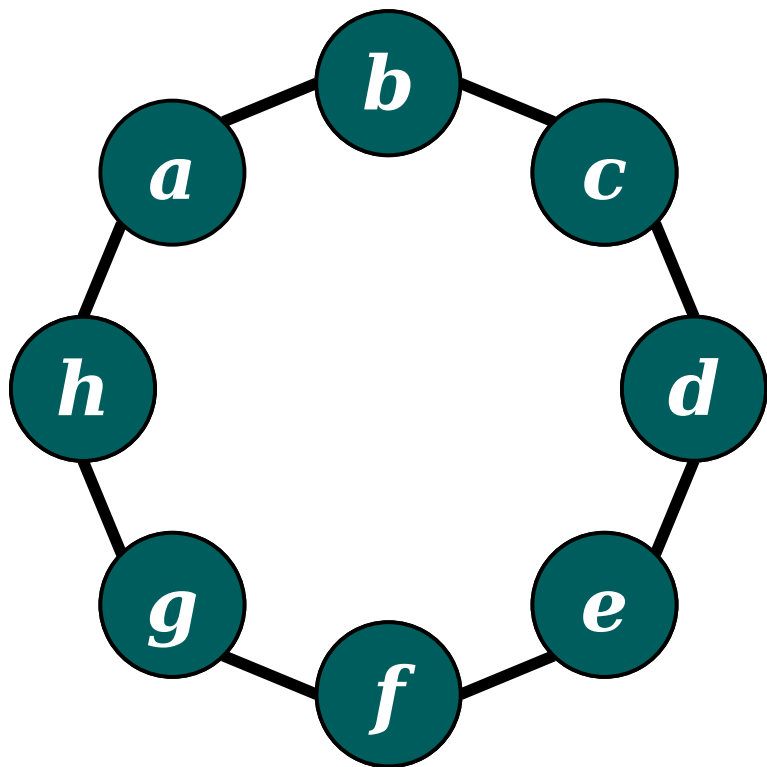
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Make a and c true.
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In Pseudocode

```
bool canTurnLightsOff(LightRing r) {  
    return isSatisfiable(ringToFormula(r));  
}
```

Intuition:

Solving Lights Out can't be “harder” than solving SAT because if we can solve SAT efficiently, we can solve Lights Out efficiently.

```
bool canPlaceDominoes(Grid G, int k) {  
    return hasMatching(gridToGraph(G), k);  
}
```

```
bool canTurnLightsOff(LightRing r) {  
    return isSatisfiable(ringToFormula(r));  
}
```

```
bool solveProblemA(string input) {  
    return solveProblemB(translate(input));  
}
```

Intuition:

Problem A can't be “harder” than problem B , because solving problem B lets us solve problem A .

```
bool solveProblemA(string input) {  
    return solveProblemB(translate(input));  
}
```

- If A and B are problems where it's possible to solve problem A using the strategy shown above*, we write

$$A \leq_p B.$$

- We say that ***A is polynomial-time reducible to B .***

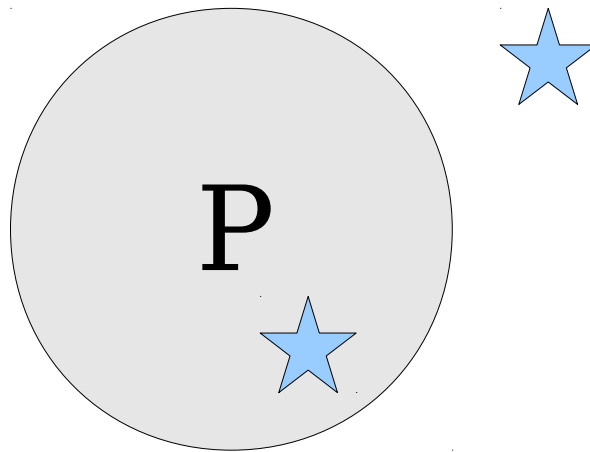
* Assuming that `translate` runs in polynomial time.

```
bool solveProblemA(string input) {  
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```

- This is a powerful general problem-solving technique. You'll see it a lot in CS161.

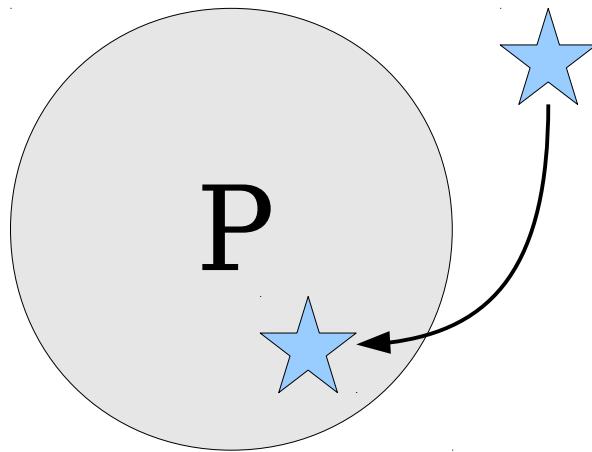
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.



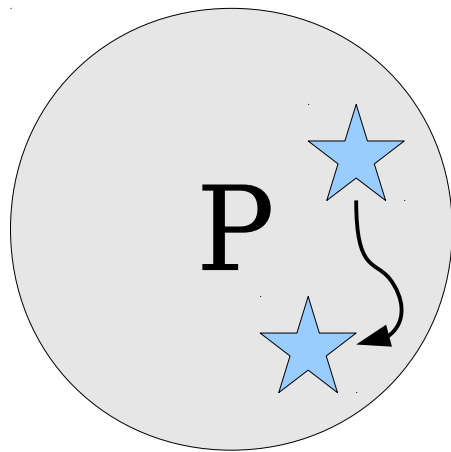
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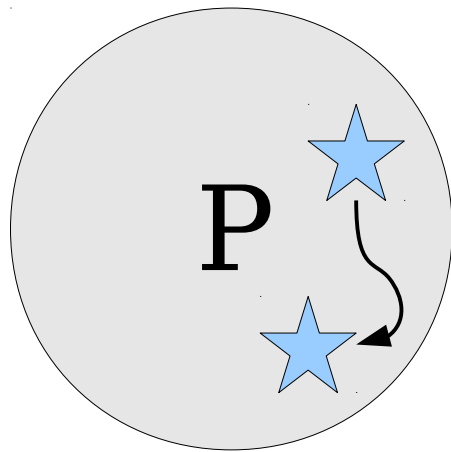
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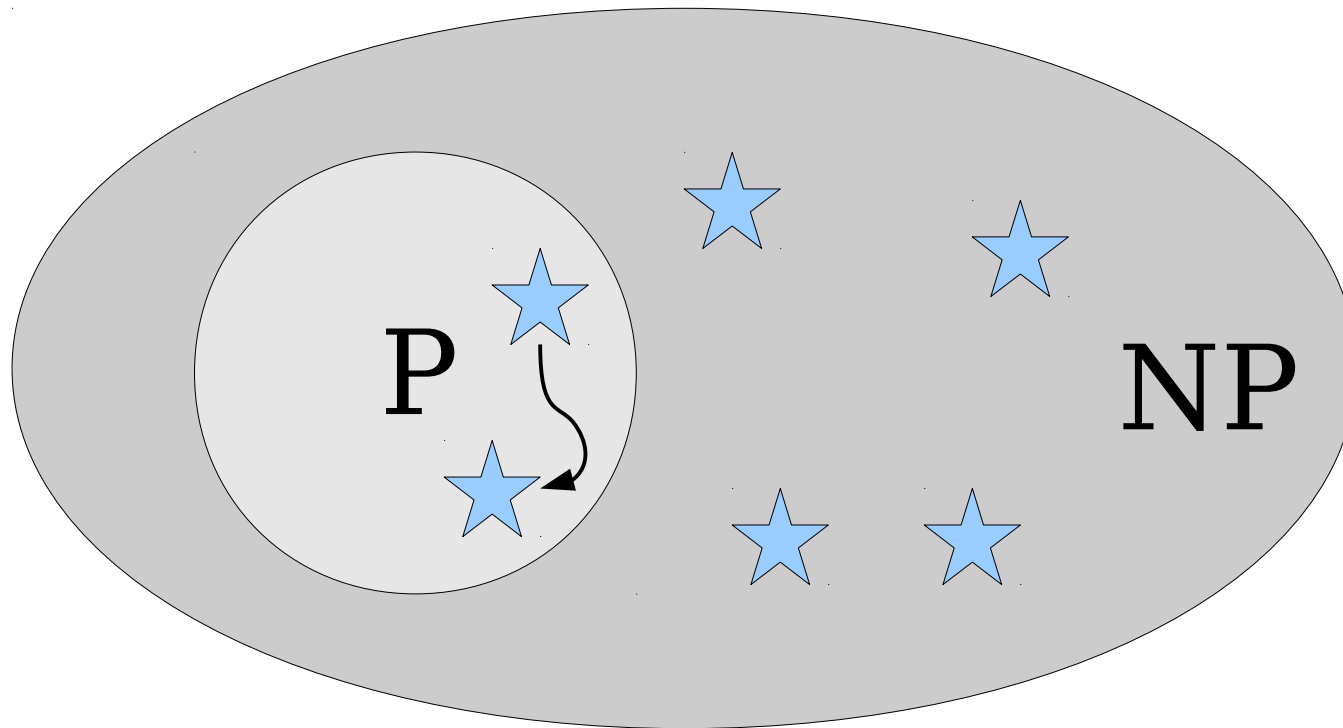
Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
- If $A \leq_p B$ and $B \in \mathbf{NP}$, then $A \in \mathbf{NP}$.



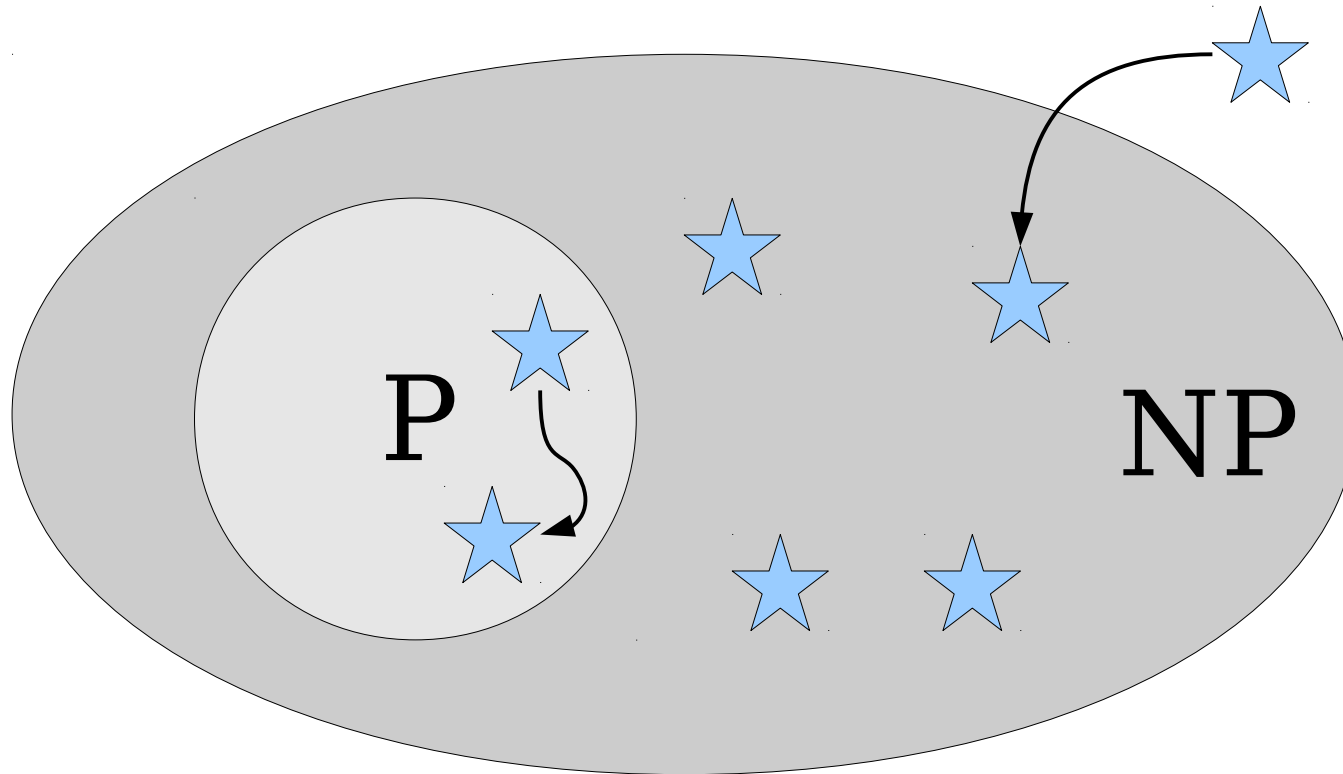
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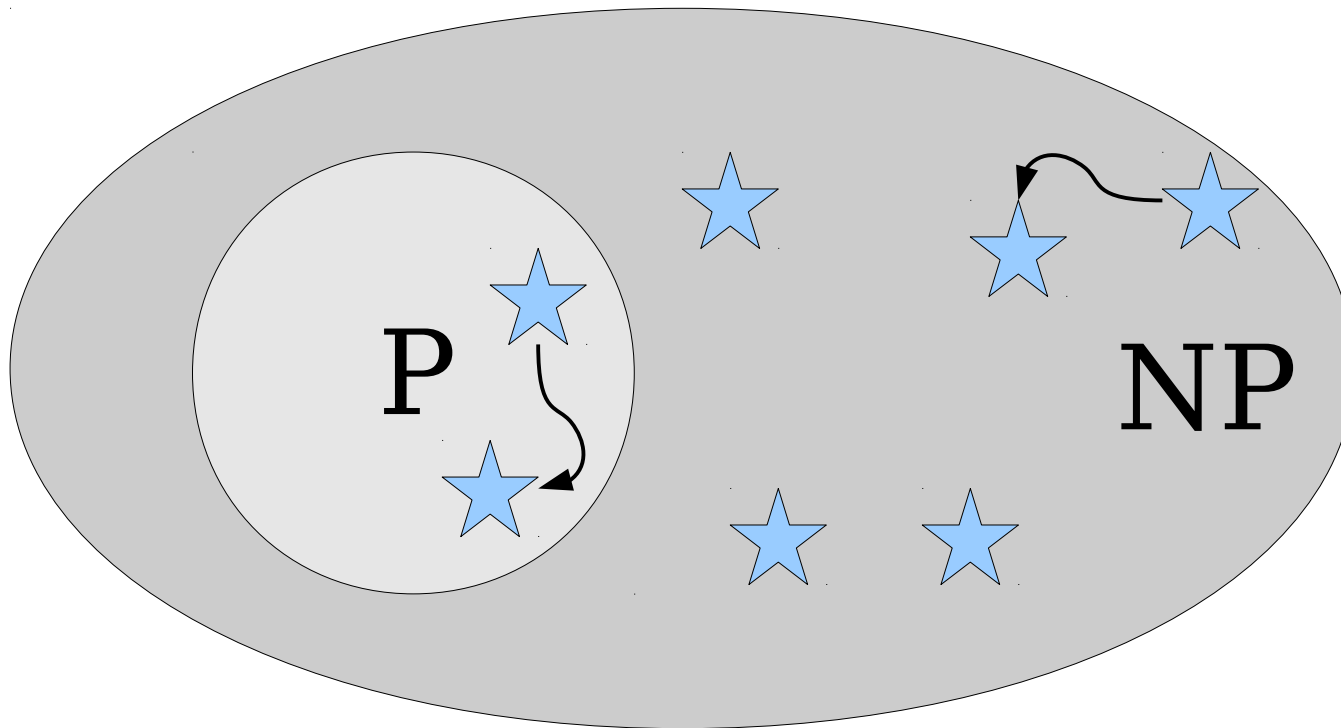
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Polynomial-Time Reductions

- If $A \leq_p B$ and $B \in \mathbf{P}$, then $A \in \mathbf{P}$.
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This $\leq_p$ relation lets us rank the relative difficulties of problems in **P** and **NP**.

What else can we do with it?

Time-Out for Announcements!

Please evaluate this course on Axess.

Your feedback makes a difference.

Don Knuth Lecture

- Don Knuth, a living legend in CS, is giving his 28th annual Christmas Lecture tomorrow (***Thursday, December 5th***) at ***5PM*** in ***Nvidia Auditorium***.
- The topic is “Strong Components and Weak Components” and a famous algorithm for finding them.
- If you’ve taken CS106B and have seen depth-first search, you know everything you need to understand this talk.
- Highly recommended!

Final Exam Logistics

- Our final exam is on ***Wednesday, December 11th*** from ***3:30PM - 6:30PM***.
 - Seating assignments will be online soon; we'll make an announcement when they're ready.
- The final exam is cumulative, covering topics from PS0 – PS9 and L00 – L26. The format is similar to that of the midterms, with a mix of short-answer questions and formal written proofs.
- Like the midterms, it's closed-book, closed-computer, and limited-note. You can bring one double-sided 8.5" × 11" notes sheet with you.

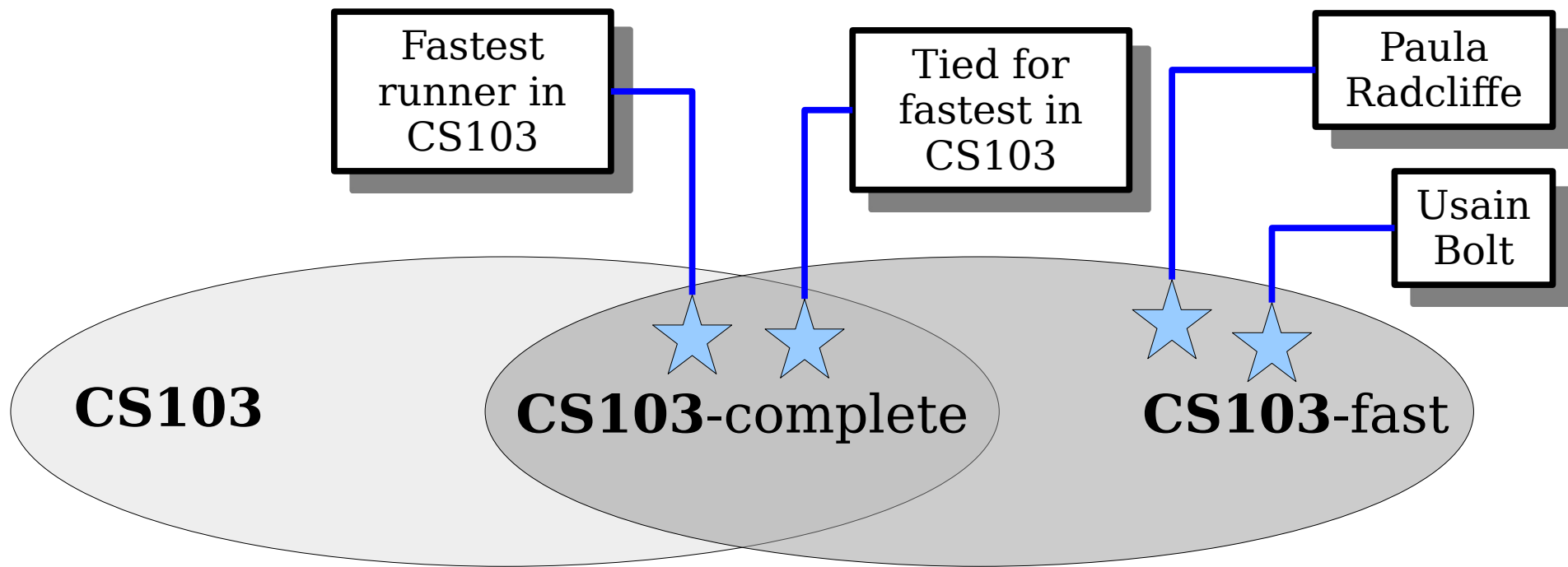
Preparing for the Exam

- Stanley will be holding a review session this Friday from 4PM – 5PM, location TBA.
- We've also released EPP3, a collection of four practice final exams you can use to prepare.
- We've also released the Cumulative Practice Problems list, a gigantic searchable database of problems you can use to brush up on whatever topics you need the most practice with.
- As always, ***keep the TAs in the loop when studying!*** That's what we're here for.

Back to CS103!

NP-Hardness and **NP**-Completeness

An Analogy: Running Really Fast



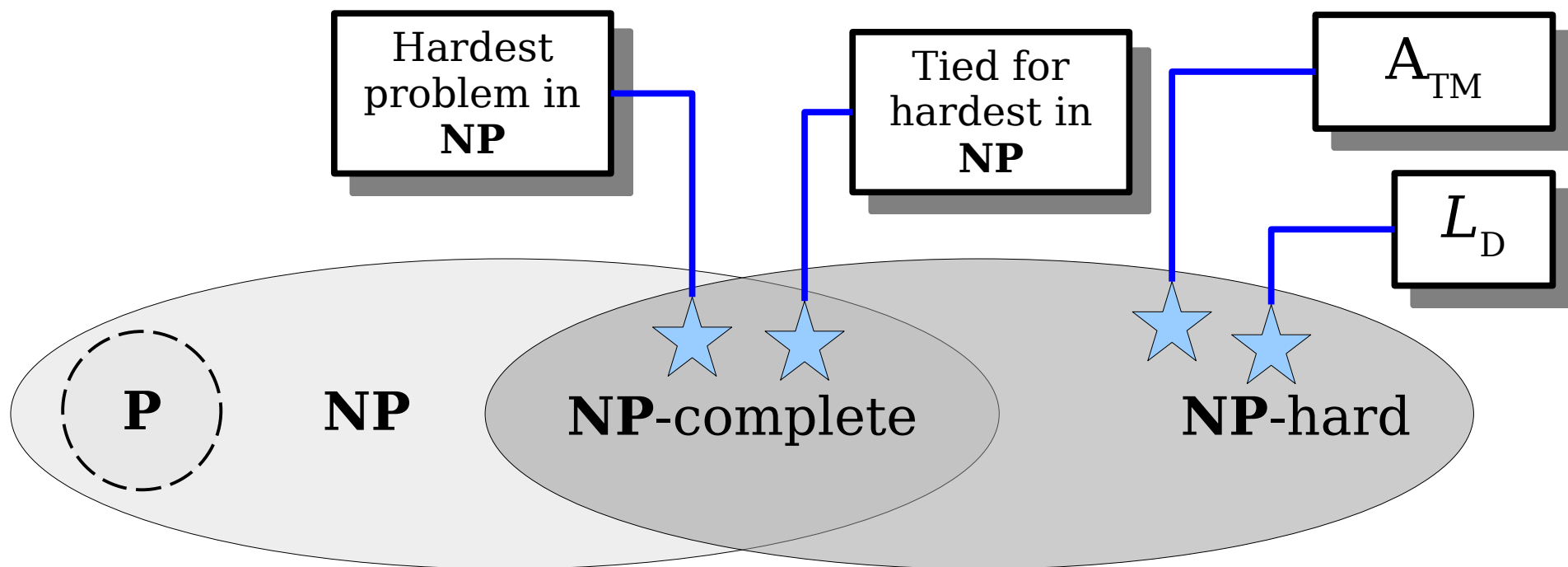
For people A and B , we say $A \leq_r B$ if A 's top running speed is at most B 's top speed.
(Intuitively: B can run at least as fast as A .)

We say that person P is **CS103-fast** if
 $\forall A \in \text{CS103}. A \leq_r P.$

(How fast are you if you're CS103-fast?)

We say that person P is **CS103-complete** if
 $P \in \text{CS103}$ and P is **CS103-fast**.

(How fast are you if you're CS103-complete?)



For languages A and B , we say $A \leq_p B$ if A reduces to B in polynomial time.

(Intuitively: B is at least as hard as A .)

We say that a language L is **NP-hard** if

$$\forall A \in \mathbf{NP}. A \leq_p L.$$

(How hard is a problem that's NP-hard?)

We say that a language L is **NP-complete** if

$$L \in \mathbf{NP} \text{ and } L \text{ is } \mathbf{NP}\text{-hard}.$$

(How hard is a problem that's NP-complete?)

Intuition: The **NP**-complete problems are the hardest problems in **NP**.

If we can determine how hard those problems are, it would tell us a lot about the **P** $\stackrel{?}{=}$ **NP** question.

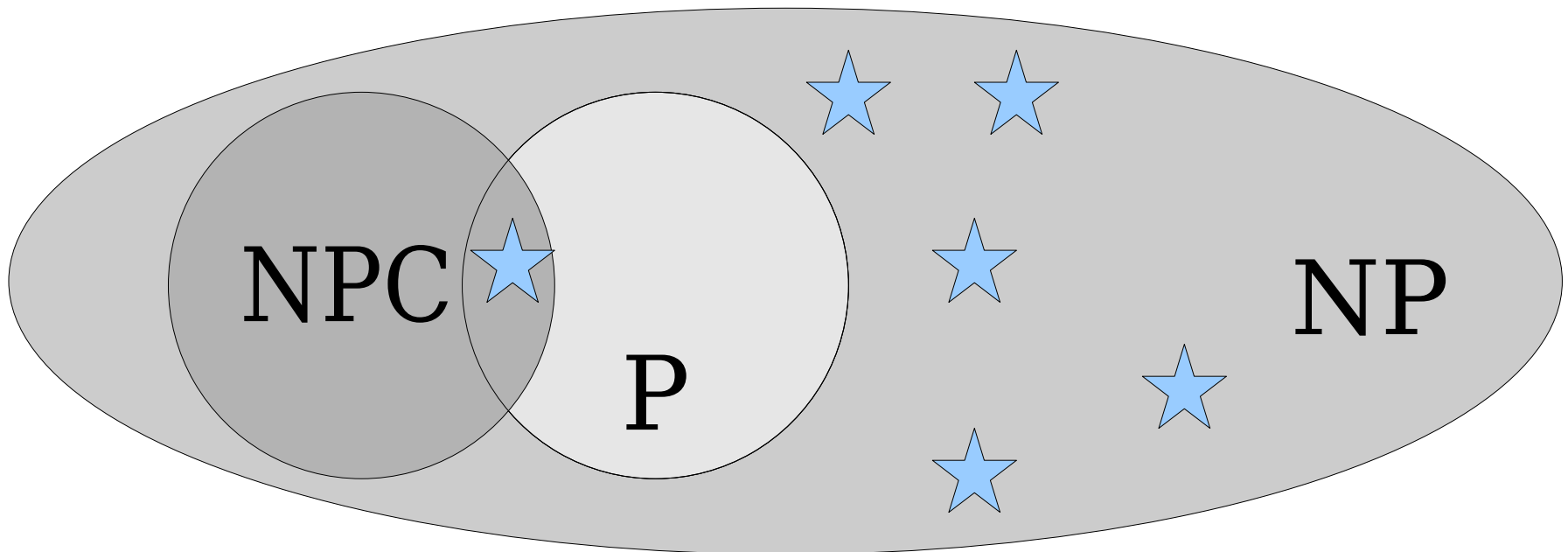
The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Intuition: This means the hardest problems in **NP** aren't actually that hard. We can solve them in polynomial time. So that means we can solve all problems in **NP** in polynomial time.

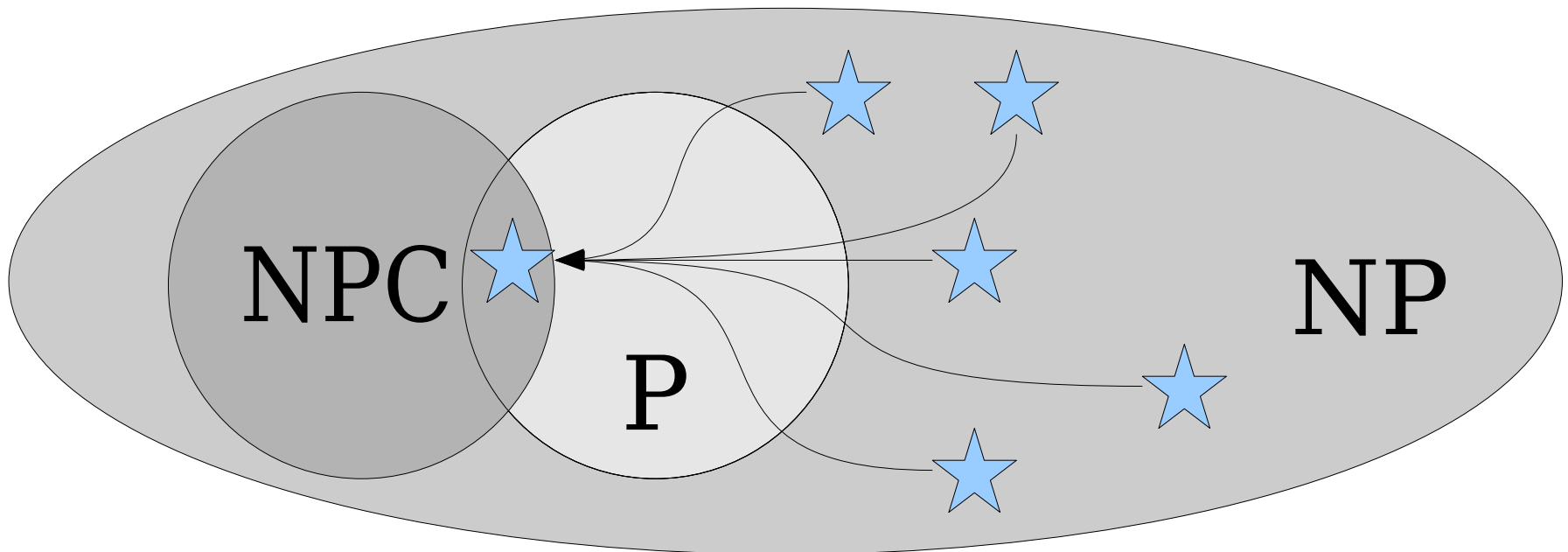
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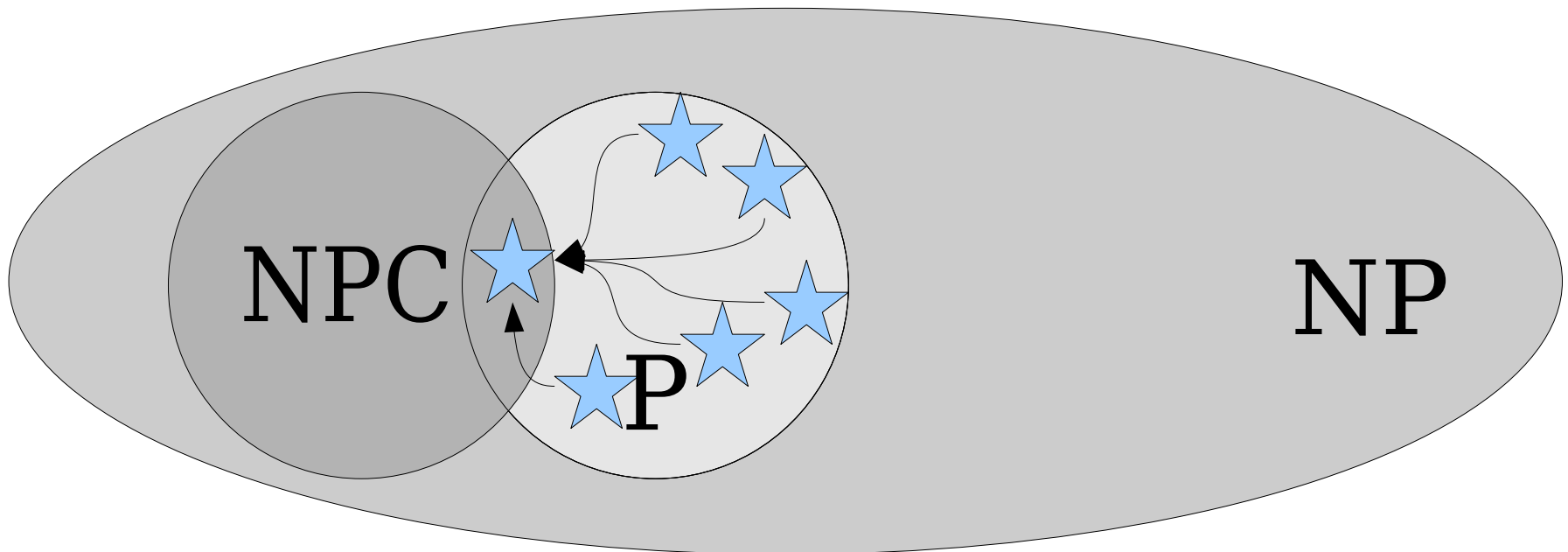
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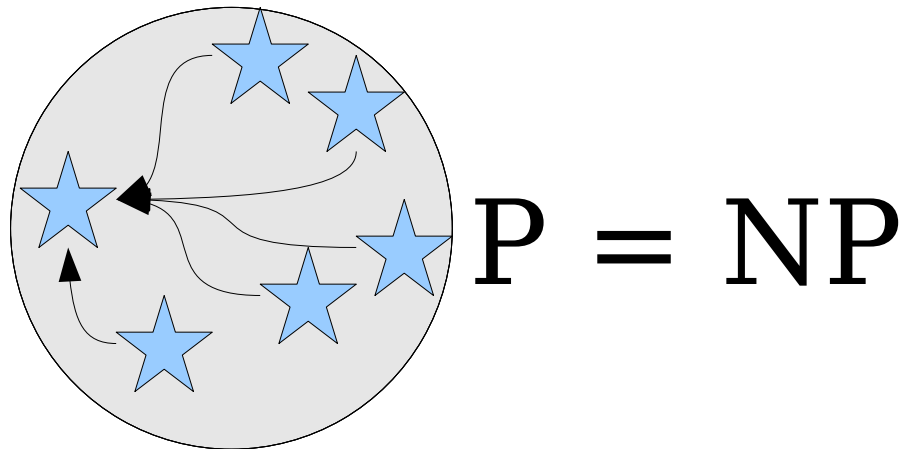
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The Tantalizing Truth

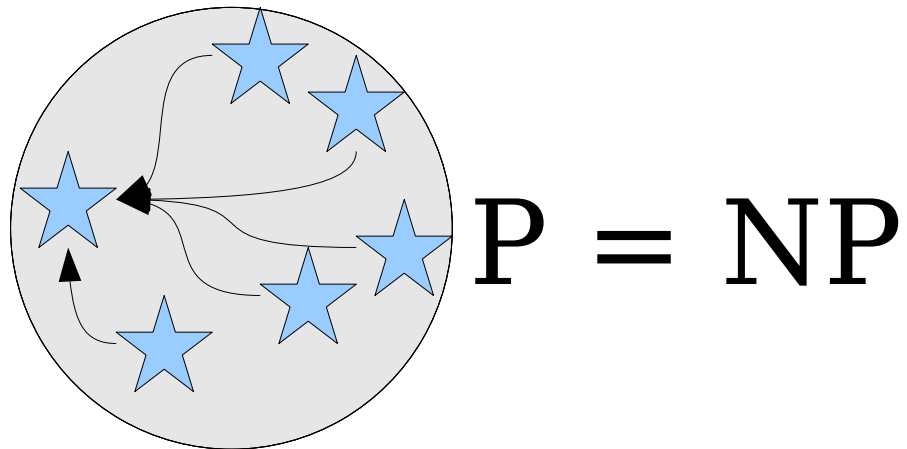
Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.



The Tantalizing Truth

Theorem: If *any* **NP**-complete language is in **P**, then **P** = **NP**.

Proof: Suppose that L is **NP**-complete and $L \in \mathbf{P}$. Now consider any arbitrary **NP** problem A . Since L is **NP**-complete, we know that $A \leq_p L$. Since $L \in \mathbf{P}$ and $A \leq_p L$, we see that $A \in \mathbf{P}$. Since our choice of A was arbitrary, this means that **NP** $\subseteq$ **P**, so **P** = **NP**. ■



The Tantalizing Truth

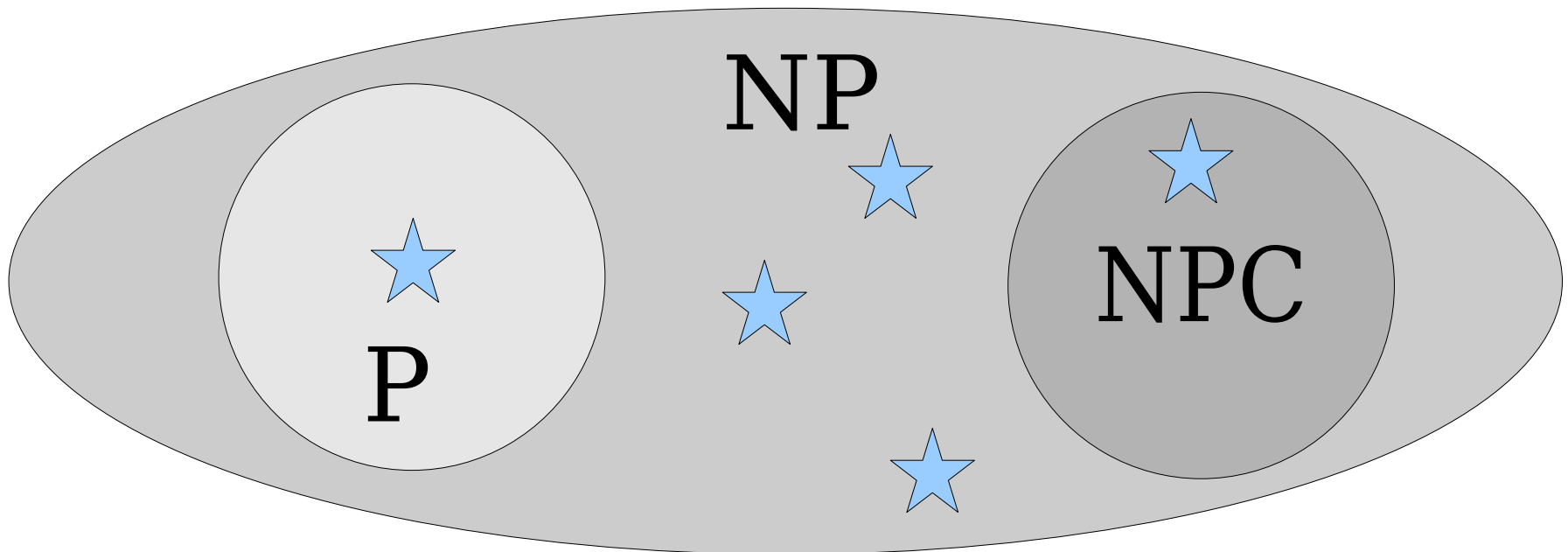
Theorem: If *any* **NP**-complete language is not in **P**, then $\mathbf{P} \neq \mathbf{NP}$.

Intuition: This means the hardest problems in **NP** are so hard that they can't be solved in polynomial time. So the hardest problems in **NP** aren't in **P**, meaning $\mathbf{P} \neq \mathbf{NP}$.

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is not in **P**, then $\mathbf{P} \neq \mathbf{NP}$.

Proof: Suppose that L is an **NP**-complete language not in **P**. Since L is **NP**-complete, we know that $L \in \mathbf{NP}$. Therefore, we know that $L \in \mathbf{NP}$ and $L \notin \mathbf{P}$, so $\mathbf{P} \neq \mathbf{NP}$. ■



How do we even know NP-complete problems exist in the first place?

Theorem (Cook-Levin): SAT is **NP**-complete.

Proof Idea: To see that **SAT** $\in$ **NP**, show how to make a polynomial-time verifier for it. Key idea: have the certificate be a satisfying assignment.

To show that **SAT** is **NP**-hard, given a polynomial-time verifier V for an arbitrary **NP** language L , for any string w you can construct a polynomially-sized formula $\varphi(w)$ that says “there is a certificate c where V accepts $\langle w, c \rangle$.” This formula is satisfiable if and only if $w \in L$, so deciding whether the formula is satisfiable decides whether w is in L . ■

Proof: Take CS154!

Why All This Matters

- Resolving $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is equivalent to just figuring out how hard SAT is.

$$\text{SAT} \in \mathbf{P} \quad \leftrightarrow \quad \mathbf{P} = \mathbf{NP}$$

- We've turned a huge, abstract, theoretical problem about solving problems versus checking solutions into the concrete task of seeing how hard one problem is.
- You can get a sense for how little we know about algorithms and computation given that we can't yet answer this question!

Why All This Matters

- You will almost certainly encounter **NP**-hard problems in practice – they're everywhere!
- If a problem is **NP**-hard, then there is no known algorithm for that problem that
 - is efficient on all inputs,
 - always gives back the right answer, and
 - runs deterministically.
- ***Useful intuition:*** If you need to solve an **NP**-hard problem, you will either need to settle for an approximate answer, an answer that's likely but not necessarily right, or have to work on really small inputs.

Sample NP-Hard Problems

- **Computational biology:** Given a set of genomes, what is the most probable evolutionary tree that would give rise to those genomes? (*Maximum parsimony problem*)
- **Game theory:** Given an arbitrary perfect-information, finite, two-player game, who wins? (*Generalized geography problem*)
- **Operations research:** Given a set of jobs and workers who can perform those tasks in parallel, can you complete all the jobs within some time bound? (*Job scheduling problem*)
- **Machine learning:** Given a set of data, find the simplest way of modeling the statistical patterns in that data. (*Bayesian network inference problem*)
- **Medicine:** Given a group of people who need kidneys and a group of kidney donors, find the maximum number of people who can receive transplants. (*Cycle cover problem*)
- **Systems:** Given a set of processes and a number of processors, find the optimal way to assign those tasks so that they complete as soon as possible. (*Processor scheduling problem*)

Coda: What if $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ is resolved?

Next Time

- ***Why All This Matters***
- ***Where to Go from Here***
- ***A Final “Your Questions”***
- ***Parting Words!***